PHYSICS 489/1489

LECTURE 7: QUANTUM ELECTRODYNAMICS

REMINDER

- Problem set 1 due today
  - 1700 in Box F

TODAY:

- We investigated the Dirac equation
  - it describes a relativistic spin 1/2 particle
  - implies the existence of antiparticle states
  - derived plane wave states for the equation
- Today we explore the electromagnetic field
  - wave equation, polarization
  - how a charged Dirac particle interacts with the field
- Recall the Golden rule
  - Rate ~ phase space x $|\text{matrix element}|^2$
  - We know how to deal with the phase space
  - today we see how to derive the matrix element

APOLOGIES:

- This will contain (hopefully) the sketchiest "derivations" and discussion of material
- The book does some detailed handwaving to derive the matrix element
  - I will handwave over the handwaving
  - A proper treatment requires quantum field theory
- For the rest of the class, the important thing to get out of this are the Feynman rules which give us a systematic way of determining the amplitude
  - deriving the Feynman rules is out of the scope of our class

THE PHOTON

- Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \quad \nabla \times \mathbf{B} - \dot{\mathbf{E}} = 4\pi\mathbf{J}$$

  - Recall that we can re-express the Maxwell equations using potentials:

$$\mathbf{E} = -\nabla\phi \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

  - these can in turn be combined to make a 4 vector:
  - Likewise for the "source" terms ρ and **J:**

$$A^\mu = (\phi, \mathbf{A})$$

$$J^\mu = (\rho, \mathbf{J})$$

MAXWELL'S EQUATION

- All four equations can be expressed as:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- The issue is that A is (far) from unique:
  - Consider: $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$

$$\partial_\mu \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu(\partial_\mu (A^\mu + \partial^\mu \lambda) =$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu (A^\mu) + \partial_\mu \partial^\mu \partial^\nu \lambda - \partial^\nu \partial_\mu \partial^\mu \lambda$$

  - last terms cancel, so "new" A_μ is also a solution to Maxwell's equation
  - they are physically the same, so we can make some conventions:
- "Lorentz gauge condition": $\partial_\mu A^\mu = 0$ $\quad \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$
- "Coulomb gauge" $\quad A^0 = 0 \quad \nabla \cdot \mathbf{A} = 0$

"FREE" SOLUTIONS

- "Free" means no sources (charges, currents): $J^{\mu}=0$
- Find solution by ansatz:

$$\partial^\mu \partial_\mu A^\nu = 0$$

$$A^\mu(x) = a\, e^{-ip\cdot x} \epsilon^\mu(p)$$

- Now check:

$$\partial_\mu A^\nu(x) = -ip_\mu\, a\, e^{-ip\cdot x} \epsilon^\nu(p)$$

$$\partial^\mu \partial_\mu A^\nu(x) = (-i)^2 p^\mu p_\mu\, a\, e^{-ip\cdot x} \epsilon^\nu(p) = 0$$

$$\partial_\mu A^\mu = 0 \Rightarrow p_\mu \epsilon^\mu(p) = 0$$

$$p^2 = m^2 c^2 = 0$$

$$A^0 = 0 \Rightarrow \epsilon^0 = 0$$

$$\Rightarrow \mathbf{p} \cdot \epsilon = 0$$

- Conclusions:
  - Photon is massless
  - Polarization ε is transverse to photon direction (in Coulomb gauge):
    - it has two degrees of freedom/polarizations

REMINDER OF DIRAC SPINORS

- Plane wave states

electrons

"positive" energy solutions

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

$$v_2 \equiv u_3 = \sqrt{E+m}\begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix} \qquad v_1 \equiv u_4 = \sqrt{E+m}\begin{pmatrix} (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

"negative" energy solutions

positrons

TOWARDS AMPLITUDES:

$$T_{fi} = \langle \phi_f | V | \phi_i \rangle + \sum_{j \neq i} \frac{\langle \phi_f | V | \phi_j \rangle \langle \phi_j | V | \phi_i \rangle}{E_i - E_j}$$

- Previously, we saw how transitions ("interactions") in the Born approximation are related to matrix elements
- Consider the electromagnetic interaction between two charged particles
- We can guess that:
  - ϕ_i, ϕ_f describe the initial and final states of these particles
    - for plane waves, these are the u, v spinors we described before.
  - V describes the interaction of these particles with the electromagnetic field (i.e. the photon).
    - this will involve the polarization vector for the photon

E&M WITH CHARGED DIRAC PARTICLES

$$H_0 = \gamma^0(\vec{\gamma} \cdot \mathbf{p} + m) \qquad V = q\gamma^0\gamma^\mu A_\mu \qquad A^\mu = (\phi, \mathbf{A})$$

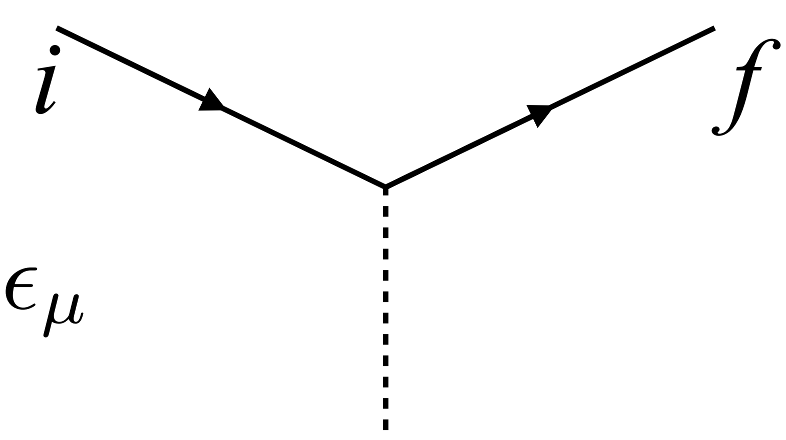
- Can be derived from “minimal substitution”

$$\partial_\mu \Rightarrow \partial_\mu + iqA_\mu \qquad p_\mu \Rightarrow p_\mu - qA_\mu$$

  - we'll see later that this can arise from “gauge invariance”
  - for now, you can see:

$$E \Rightarrow E - q\phi \qquad \mathbf{p} \Rightarrow \mathbf{p} - q\mathbf{A}$$

    - i.e. impact from scalar/vector potential on the energy of the particle
  - We can then express the interaction as follows



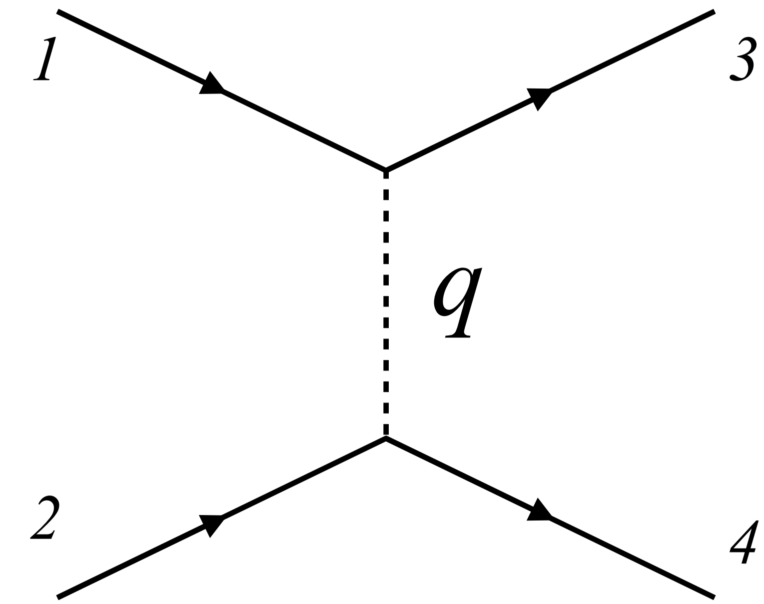
$$\langle\psi_f|V|\psi_i\rangle \Rightarrow u_f^\dagger\, Q\gamma^0\gamma^\mu\epsilon_i\, u_i \Rightarrow Q\, \bar{u}_f\gamma^\mu u_1\, \epsilon_\mu$$

INTERACTION BETWEEN TWO CHARGED PARTICLES VIA EM FIELD

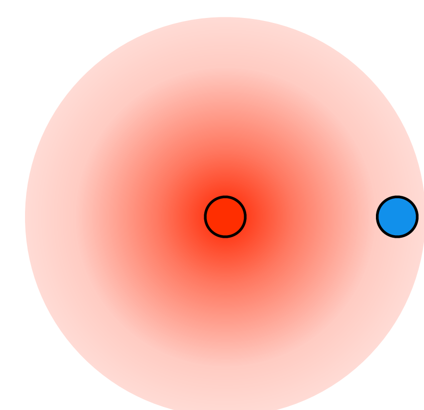
$$\langle\psi_3|V|\psi_1\rangle \Rightarrow Q\ \bar{u}_3\gamma^\mu u_1\ \epsilon_\mu$$

$$\langle\psi_4|V|\psi_2\rangle \Rightarrow Q\ \bar{u}_4\gamma^\nu u_2\ \epsilon_\nu$$

$$\frac{1}{q^2}$$



$$\sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)*} = -g_{\mu\nu}$$

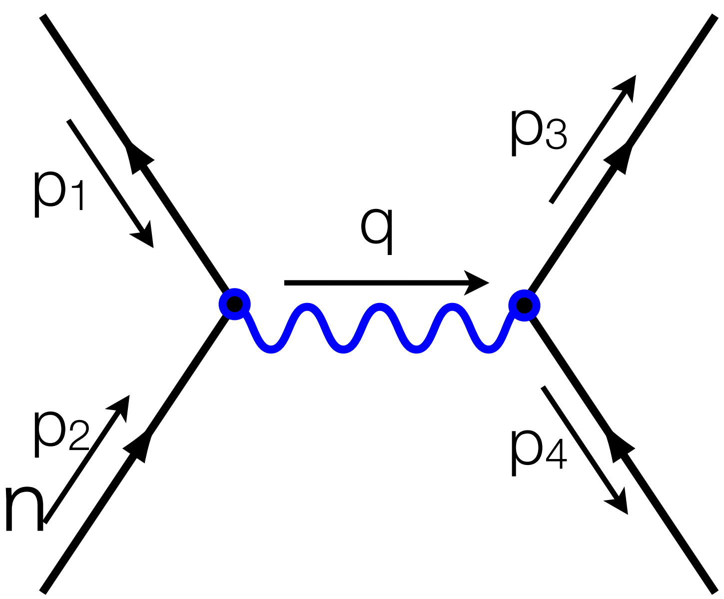


- "Propagator" factor for photon
  - reflects potential energy of the field
- Sum over polarizations of intermediate photon
  - note that this results in "contracting" the Lorentz indices
- Accounts for both "orderings" of the photon propagation

$$\mathcal{M} = \frac{-Q^2}{q^2}\left[\bar{u}_3\gamma^\mu u_1\right]\left[\bar{u}_4\gamma_\mu u_2\right]$$

FEYNMAN RULES: EXTERNAL LINES

- Right down the Feynman diagram(s) for the process and label the momentum flow
  - p's for external lines, q's for internal lines
  - Note that there are two flows:
    - "particle/antiparticle"
    - momentum



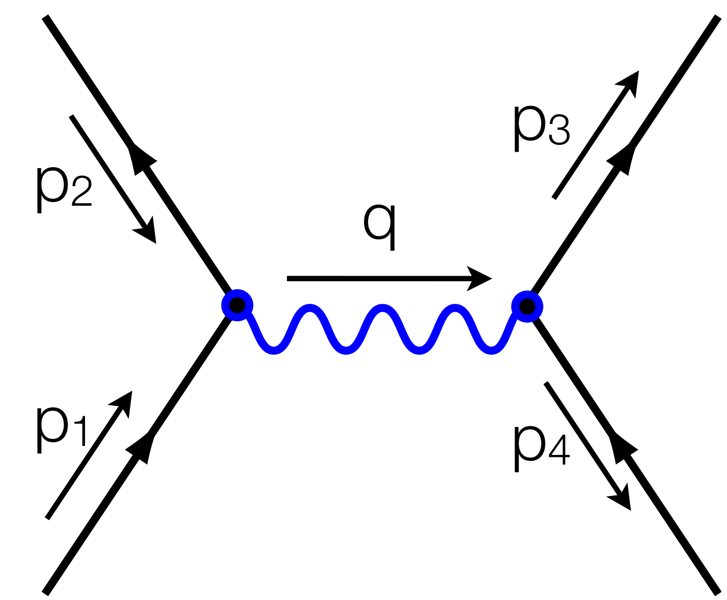
- Now the components of the expression
  - External Lines:
    - Electrons: initial state $u^s(p)$ final state $\bar{u}^s(p)$
    - Positrons: initial state $\bar{v}^s(p)$ final state $v^s(p)$
    - Photons: initial state $\epsilon_\mu(p)$ final state $\epsilon^*_\mu(p)$

VERTICES AND PROPAGATORS

- For each QED vertex: $ig_e\gamma^\mu \;\; (2\pi)^4\delta^4(k_1 + k_2 + k_3)$

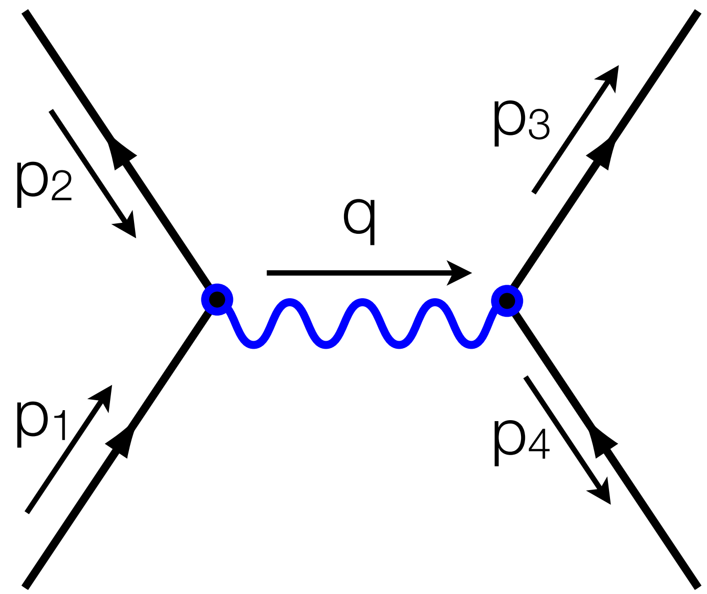
use a different Lorentz index for each γ matrix

  - momentum is "+" incoming, "-" outgoing from vertex
  - g_e is the electromagnetic coupling (Qe)
- Internal lines:
  - electron/positron propagator $\dfrac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$
  - Photon propagator $\dfrac{-ig_{\mu\nu}}{q^2}$
    - indices match vertices/polarization
  - Integral over internal momentum: $\dfrac{d^4q}{(2\pi)^4}$



- Finally: cancel the overall $(2\pi)^4\,\delta^4$ what remains is -iM

EXAMPLE: $e^+ + e^- \rightarrow \mu^+ + \mu^-$



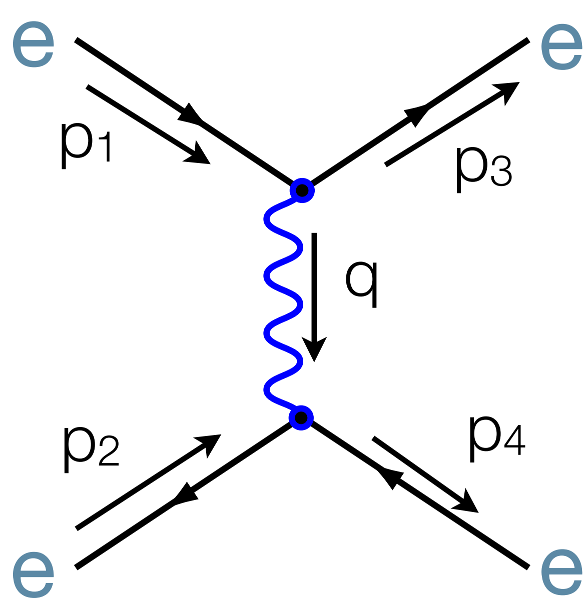
- Order matters due to Dirac matrix structure (photon part doesn't care)
- Go backward along the fermion lines:
  - In the "final state": $\bar{u}(3)\; i g_e \gamma^\mu \; v(4) \quad (2\pi)^4 \delta^4(q - p_3 - p_4)$
  - In the "initial state": $\bar{v}(2)\; i g_e \gamma^\nu \; u(1) \quad (2\pi)^4 \delta^4(p_1 + p_2 - q)$
  - Throw in the internal photon propagator: $\frac{1}{(2\pi)^4} \int d^4 q \, \frac{-i g_{\mu\nu}}{q^2}$
  - integrate over internal momentum

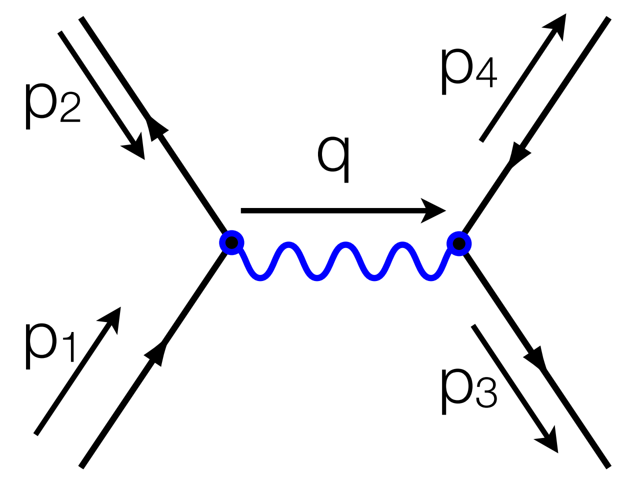
$$i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times \frac{g_e^2}{(p_1 + p_2)^2} \, [\bar{u}(3)\; \gamma^\mu \; v(4)] \; g_{\mu\nu} \; [\bar{v}(2)\; \gamma^\nu \; u(1)]$$

Problem 6.5

$$\mathcal{M} = -\frac{g_e^2}{(p_1 + p_2)^2} \, [\bar{u}(3)\; \gamma^\mu \; v(4)] \, [\bar{v}(2)\; \gamma_\mu \; u(1)]$$

EXAMPLE: $e^+ + e^- \rightarrow e^+ + e^-$





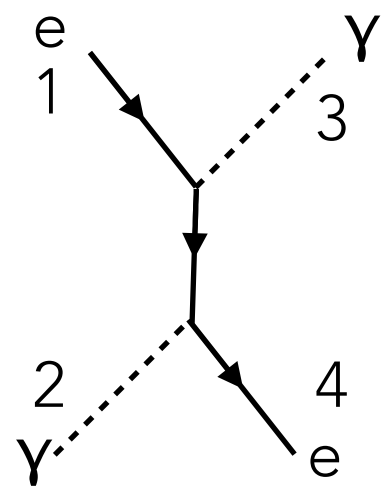
$$\bar{u}(3)\ ig_e\gamma^\mu\ u(1)\ \ \bar{v}(2)\ ig_e\gamma^\nu\ v(4)\ \ \frac{-ig_{\mu\nu}}{(p_1-p_3)^2}$$

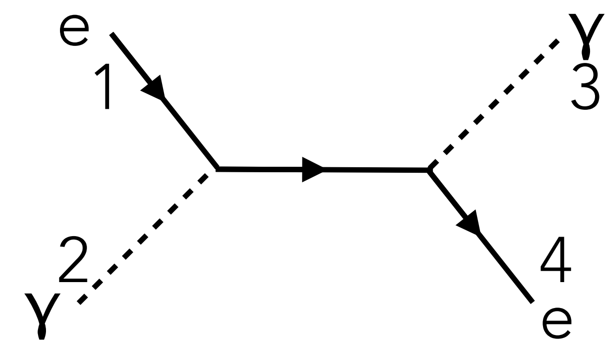
$$\bar{u}(3)\ ig_e\gamma^\rho\ v(4)\ \ \bar{v}(2)\ ig_e\gamma^\sigma\ u(1)\ \ \frac{-ig_{\rho\sigma}}{(p_1+p_2)^2}$$

$$(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$$

A FEW MORE:

- $e^- + \gamma \rightarrow e^- + \gamma$

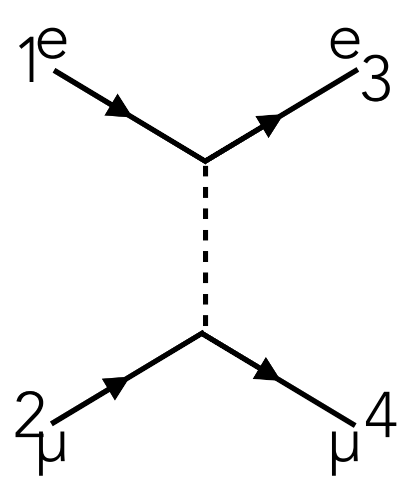




$$\mathcal{M}_1 = \frac{g_e^2}{(p_1 - p_3)^2 - m^2}\left[\bar{u}_4 \not{\epsilon}_2 (\not{p}_1 - \not{p}_3 + m)\not{\epsilon}_3^* u_1\right]$$

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1 + p_2)^2 - m^2}\left[\bar{u}_4 \not{\epsilon}_3^* (\not{p}_1 + \not{p}_2 + m)\not{\epsilon}_2 u_1\right]$$

- $e^- + \mu^- \rightarrow e^- + \mu^-$



$$\mathcal{M} = \frac{-g_e^2}{(p_1 - p_3)^2}\left[\bar{u}_3 \gamma^\mu u_1\right]\left[\bar{u}_4 \gamma_\mu u_2\right]$$

NEXT TIME:

- Please read 6.1 and 6.2
  - I would explicitly work out spin summation procedure in 6.2.1 and 6.2.4