PHY489/1489

THE DIRAC EQUATION

SO FAR:

- Quick introduction to:
  - special relativity, relativistic kinematics
  - quantum mechanics, golden rule, etc.
- Phase space:
  - how to set up and integrate over phase space to determine integrated and differential rates/cross sections
- Now we move to the particles themselves
  - start with the Dirac equation
  - describes “spin 1/2” particles
    - quarks, leptons, neutrinos

RELATIVISTIC WAVE FUNCTION

- In non-relativistic quantum mechanics, we have the Schrödinger Equation:

$$\mathbf{H}\psi = i\hbar\frac{\partial}{\partial t}\psi \qquad \mathbf{H} = \frac{\mathbf{p}^2}{2m} \qquad \mathbf{p} \Leftrightarrow -i\hbar\nabla$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$$

- Inspired by this, Klein and Gordon (and actually Schrödinger) tried:

$$E^2 = \mathbf{p}^2 + m^2 \Rightarrow (-\nabla^2 + m^2)\psi = -\frac{\partial^2}{\partial t^2}\psi$$

$$\left(-\frac{\partial^2}{\partial t^2}\psi + \nabla^2\right)\psi = m^2\psi$$

"Manifestly Lorentz Invariant"

$$\partial_\mu \Rightarrow (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$(-\partial^\mu\partial_\mu + m^2)\psi = 0$$

ISSUES WITH KLEIN-GORDON

- Within the context of quantum mechanics, this had some issues:
  - As it turns out, this allows negative probability densities: $|\psi|^2 = 0$
- Dirac traced this to the fact that we had second-order time derivative
  - "factor" the E/p relation to get linear relations and obtained:

$$p_\mu p^\mu - m^2c^2 = 0 \Rightarrow (\alpha^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m)$$

- and found that:

$$\alpha^\kappa = \gamma^\kappa$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- Dirac found that these relationships could be held by matrices, and that the corresponding wave function must be a "vector".

$$\gamma^\mu p_\mu - m = 0 \Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$$

THE DIRAC EQUATION IN ITS MANY FORMS

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \not{a} \equiv a_\mu\gamma^\mu$$

$$(i\not{\partial} - m)\psi = 0 \qquad \not{a} \equiv a_\mu\gamma^\mu = a_0\gamma^0 - a_1\gamma^1 - a_2\gamma^2 - a_3\gamma^3$$

$$\partial_\mu \Rightarrow (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\left[i(\gamma^0\partial_0 - \gamma^1\partial_1 - \gamma^2\partial_2 - \gamma^3\partial_3) - m\right]\psi = 0$$

$$\left[i\left(\gamma^0\frac{\partial}{\partial t} - \gamma^1\frac{\partial}{\partial x} - \gamma^2\frac{\partial}{\partial y} - \gamma^3\frac{\partial}{\partial z}\right) - m\right]\psi = 0$$

DIRAC QUOTES:

- Q: What do you like best about America?
- A: Potatoes.

- Q: What is you favourite sport?
- A: Chinese chess.

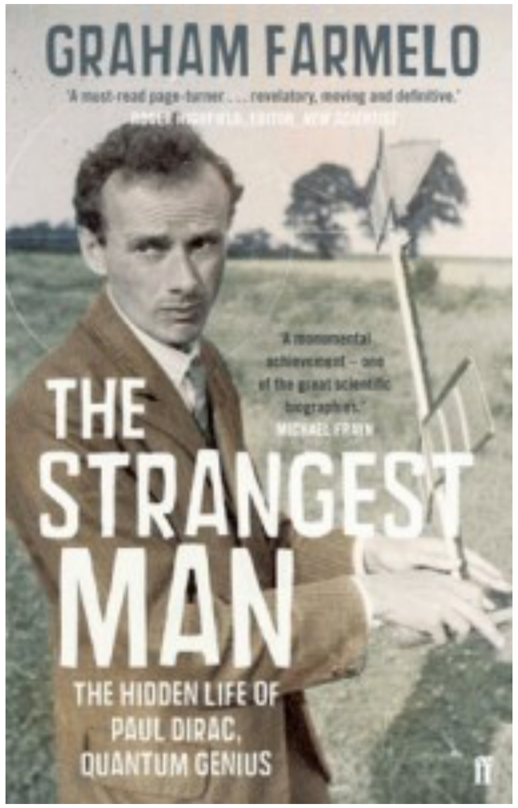
- Q: Do you go to the movies?
- A: Yes.
- Q: When?
- A: 1920, perhaps also 1930.

- Q: How did you find the Dirac Equation?
- A: I found it beautiful.

- "Dirac's spoken vocabulary consists of "yes", "no", and "I don't know."

- Q: Professor Dirac, I did not understand how you derived the formula on the top left
- A: That is not a question. It is a statement. Next question, please.

- I was taught at school never to start a sentence without knowing the end of it.



"GAMMA" MATRICES:

$$\gamma^\mu = (\gamma^0,\ \gamma^1,\ \gamma^2, \gamma^3) \qquad \vec{\sigma} = (\sigma^1,\ \sigma^2,\ \sigma^3) = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

- Note that this is a particular representation of the matrices
- Any set of matrices satisfying the anti-commutation relations works
- There are an infinite number of possibilities: this particular one (Björken-Drell) is just one example

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

IN FULL GLORY

$$\left[i \begin{pmatrix} \frac{\partial}{\partial t} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial t} & -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} & 0 \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial t} \end{pmatrix} - \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \right] \left(\begin{array}{c} \overset{\psi_A}{\boxed{\begin{matrix} \psi_1 \\ \psi_2 \end{matrix}}} \\ \underset{\psi_B}{\boxed{\begin{matrix} \psi_3 \\ \psi_4 \end{matrix}}} \end{array} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p_\mu \Leftrightarrow -\partial_\mu$$

$$\begin{pmatrix} p_0 - m & -\mathbf{p} \cdot \sigma \\ \mathbf{p} \cdot \sigma & -p_0 - m \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p_0 + m & -\mathbf{p} \cdot \sigma \\ \mathbf{p} \cdot \sigma & -p_0 + m \end{pmatrix} \begin{pmatrix} p_0 - m & -\mathbf{p} \cdot \sigma \\ \mathbf{p} \cdot \sigma & -p_0 - m \end{pmatrix} = \begin{pmatrix} p_0^2 - m^2 - (\mathbf{p} \cdot \sigma)^2 & 0 \\ 0 & p_0^2 - m^2 - (\mathbf{p} \cdot \sigma)^2 \end{pmatrix}$$

$$(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b}) \qquad (\sigma \cdot \mathbf{p})^2 = \mathbf{p} \cdot \mathbf{p}$$

$$\begin{pmatrix} p_0^2 - m^2 - \mathbf{p}^2 & 0 \\ 0 & p_0^2 - m^2 - \mathbf{p}^2 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- This is just the KG equation four times
  - Wavefunctions that satisfy the Dirac equation also satisfy KG

SOLUTIONS TO THE DIRAC EQUATION

- Particle at rest
  - General plane wave solution

$$\psi(x) \sim e^{-ik\cdot x}$$

  - A particle at rest has only time dependence.

$$(i\gamma^0 \frac{\partial}{\partial t} - mc)\psi = 0 \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t}\psi_A \\ \frac{\partial}{\partial t}\psi_B \end{pmatrix} = -im \begin{pmatrix} \frac{\partial}{\partial t}\psi_A \\ \frac{\partial}{\partial t}\psi_B \end{pmatrix}$$

  - The equation breaks up into two independent parts:

$$\frac{\partial}{\partial t}\psi_A = -im\psi_A \qquad -\frac{\partial}{\partial t}\psi_B = -im\psi_B$$

$$\psi_A = e^{-imt}\psi_A(0) \qquad \psi_B = e^{+imt}\psi_B(0)$$

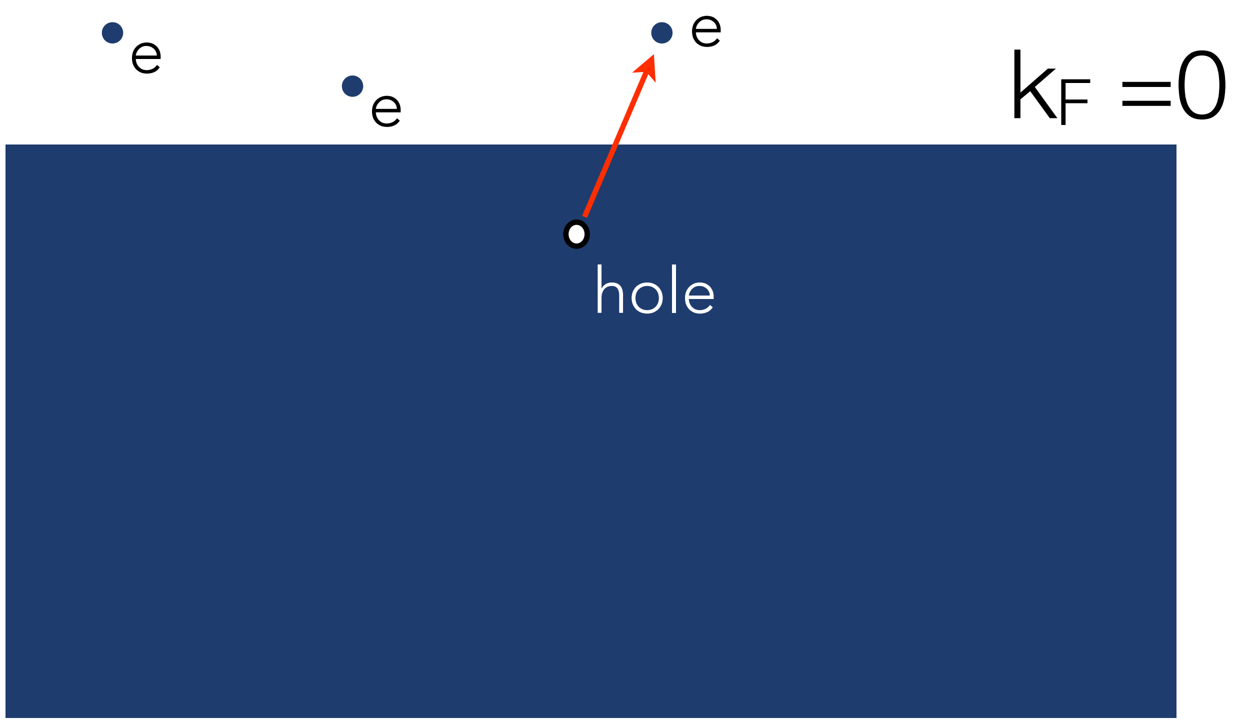
DIRAC'S DILEMMA

- ψ_B appears to have negative energy

$$\psi_A = e^{-imt}\psi_A(0) \qquad \psi_B = e^{+imt}\psi_B(0)$$

- Why don't all particles fall down into these states (and down to -∞)?

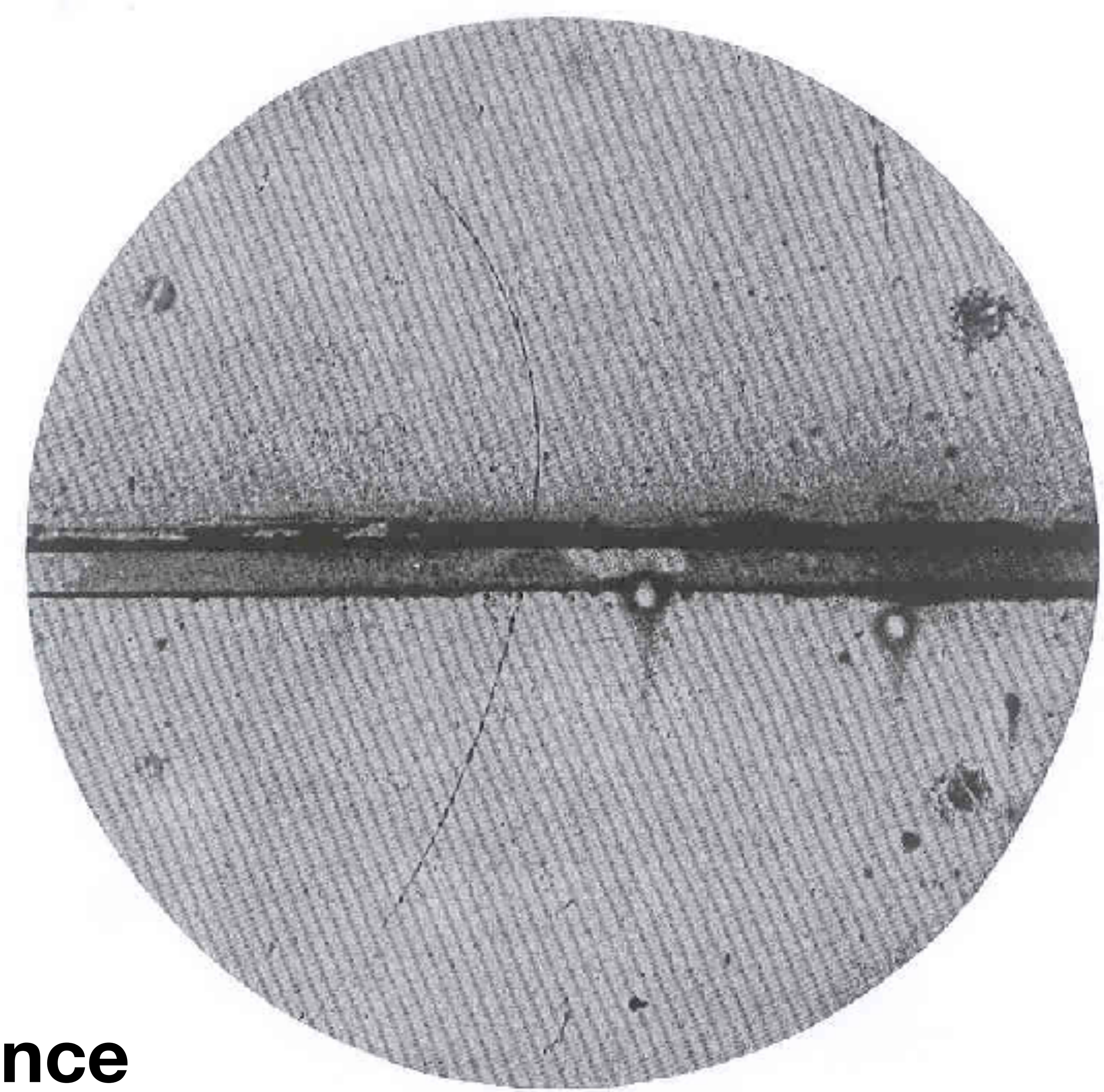
- Dirac's excuse: all states in the universe up to a certain level (say E=0) are filled.
- Pauli exclusion prevents collapse of states down to E = -∞
- We can "excite" particles out of the sea into free states This leaves a "hole" that looks like a particle with opposite properties (positive charge, opposite spin, etc.)



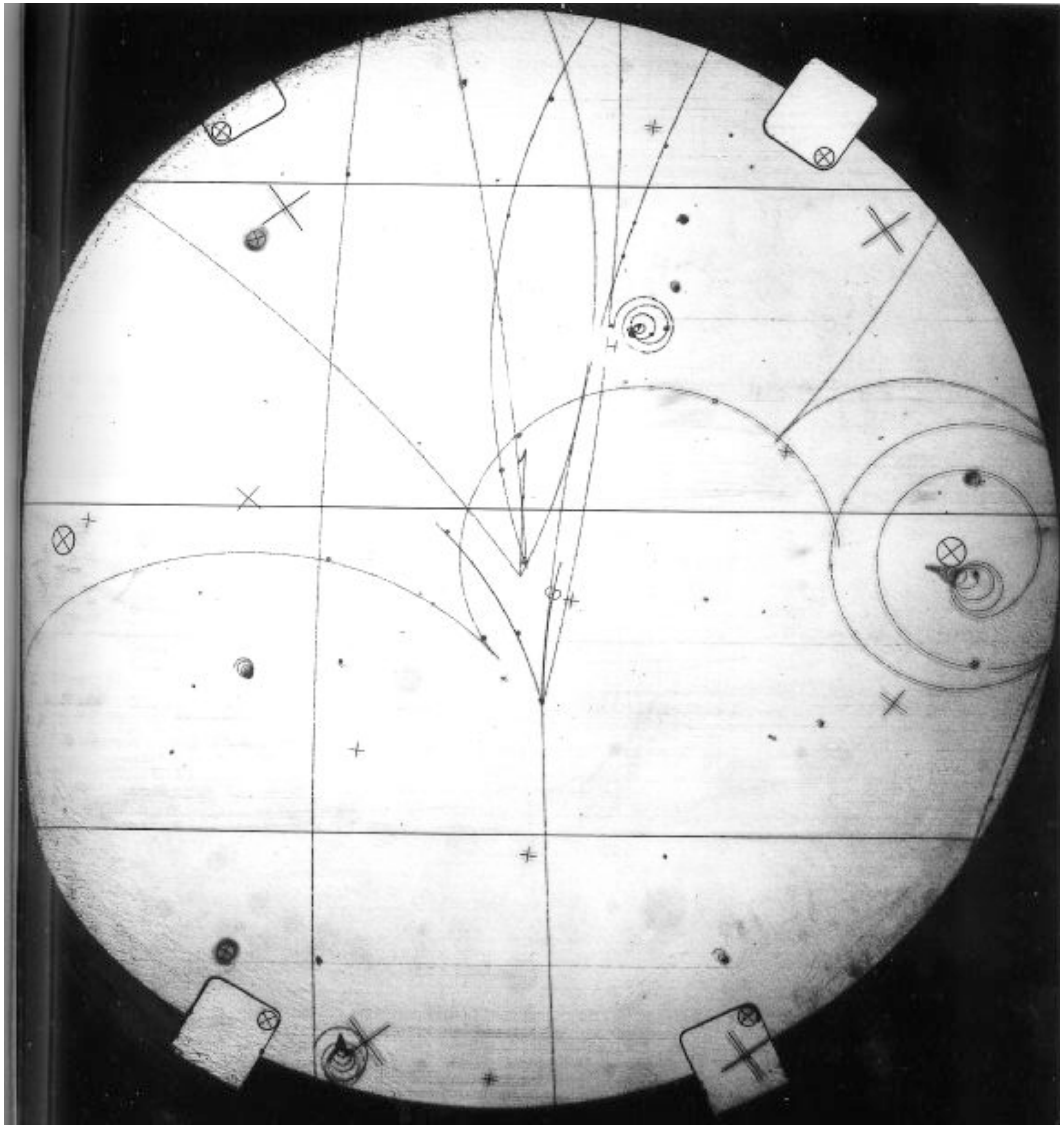
Dirac originally proposed that this might be the proton

EXCUSE TO TRIUMF

- 1932: Anderson finds "positrons" in cosmic rays
- Exactly like electrons but positively charged: Fits what Dirac was looking for



Dirac predicts the existence of anti-matter and it is found



SOLUTIONS TO THE DIRAC EQUATION

- Note that all particles have the same mass

$$\psi_1(t) = e^{-imc^2t/\hbar}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

“spin up”

$$\psi_2(t) = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

“spin down”

positive energy solutions (particle)

$$\psi_3(t) = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

“spin down”

$$\psi_4(t) = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

“spin up”

“negative” energy solutions (anti-particle)

PEDAGOCIAL SORE POINT

- Discussion we had thus far about difficulties with relativistic equations, (negative probabilities, negative energies) is historic
- The framework for dealing with quantum mechanics and special relativity (i.e. quantum field theory) had not been developed
    - The old tools of NR quantum mechanics had reached their limit and new ones were necessary.
    - In particular, the idea of a “wavefunction” had to be revisited
    - Until this was done, there were many difficulties!
    - Once QFT was developed, all of these problems go away.
- Both KG and Dirac Equations are valid in QFT
    - No negative probabilities, no negative energies

PLANE WAVE SOLUTIONS TO THE DIRAC EQUATION

- Consider a solution of the form:

$$\psi(x) = e^{-ik\cdot x}u(k) \leftarrow$$

column vector

Column vector of 4 elements with space-time dependence

space-time dependence

- and place it in the Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

$$(\gamma^{\mu}k_{\mu} - m)e^{-ik\cdot x}u(k)\psi = 0 \qquad (\gamma^{\mu}k_{\mu} - m)u(k)\psi = 0$$

MORE EXPLICITLY:

- By definition:

$$\gamma^\mu k_\mu = \gamma^0 k^0 - \gamma^1 k^1 - \gamma^2 k^2 - \gamma^3 k^3 \longrightarrow \begin{pmatrix} k_0 & -\mathbf{k}\cdot\sigma \\ \mathbf{k}\cdot\sigma & -k_0 \end{pmatrix}$$

- So that the Dirac equation reads:

Note 2x2 notation

$$(\gamma^\mu k_\mu - m)u(k) = 0$$

$$u(k) \Rightarrow \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\begin{pmatrix} k_0 - mc & -\mathbf{k}\cdot\sigma \\ \mathbf{k}\cdot\sigma & -k_0 - mc \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(k_0 - m)u_A - (\mathbf{k}\cdot\sigma)u_B = 0 \qquad u_A = \frac{\mathbf{k}\cdot\sigma}{k_0 - m}u_B$$

$$(\mathbf{k}\cdot\sigma)u_A - (k_0 + m)u_B = 0 \qquad u_B = \frac{\mathbf{k}\cdot\sigma}{k_0 + m}u_A$$

DETERMING u

- this means we can identify k ↔ ±p

$$u_A = \frac{\mathbf{k}\cdot\sigma}{k_0 - m} u_B \qquad u_B = \frac{\mathbf{k}\cdot\sigma}{k_0 + m} u_A$$

$$\frac{\mathbf{k}\cdot\sigma}{k_0 + m}\frac{\mathbf{k}\cdot\sigma}{k_0 - m} = 1$$

$$p^\mu = \pm k^\mu$$

$$\mathbf{p}\cdot\sigma = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

- We can now construct the column vector u:

electrons

Used positive k solutions

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} \qquad u_2 = N \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

positrons

Used negative k solutions

$$u_3 = N \begin{pmatrix} p_z/(E+m) \\ (p_x + ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

NORMALIZATION:

- Choose “normalization” of the wavefunctions
  - Note that multiples of the solutions are still solution
  - normalization convention simply fixes this arbitrary choice:

$$u^\dagger u = 2E \qquad u^\dagger \equiv (u^T)^* \qquad\qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad \Rightarrow \quad u^\dagger = (u_1^*,\ u_2^*,\ u_3^*\ u_4^*)$$

  - for u_1

$$u_1^\dagger u_1 = N^2 \left[1 + \frac{\mathbf{p}^2}{(E+m)^2} \right] = 2E \qquad N = \sqrt{E+m}$$

LORENTZ COVARIANCE

- The Dirac equation "works" in all reference frames.
  - What exactly does this mean?

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

"Lorentz Covariant"

- i, m, γ are constants that don't change with reference frames.
- ∂_μ and ψ will change with reference frames, however.
  - ∂_μ is a derivative that will be taken with respect to the space-time coordinates in the new reference frame. We'll call this ∂'_μ
  - how does ψ change?
    - $\psi' = S\psi$ where ψ' is the spinor in the new reference frame
- Putting this together, we have the following transformation of the equation when evaluating it in a new reference frame

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \Rightarrow (i\gamma^\mu\partial'_\mu - m)\psi' = 0$$

$$i\gamma^\mu\partial'_\mu(S\psi) - m(S\psi) = 0$$

PROPERTIES OF S

- In general, we know how to relate space time coordinates in one reference with another (i.e. Lorentz transformation), we can do the same for the derivatives
- Using the chain rule, we get: $\partial'_\mu \equiv \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}} \frac{\partial}{\partial x^\nu}$
  - where we view x as a function x' (i.e. the original coordinates as a function of the transformed or primed coordinates).
  - Note the summation over ν
- if the primed coordinates moving along the x axis with velocity β:

$$\begin{array}{lcl} x^0 & = & \gamma(x^{0\prime} + \beta x^{1\prime}) \\ x^1 & = & \gamma(x^{1\prime} + \beta x^{0\prime}) \\ x^2 & = & x^{2\prime} \\ x^3 & = & x^{3\prime} \end{array}$$

$$(\nu = 0, \mu = 0) \Rightarrow \frac{\partial x^0}{\partial x^{0\prime}} = \gamma$$

$$(\nu = 0, \mu = 1) \Rightarrow \frac{\partial x^0}{\partial x^{1\prime}} = \gamma\beta$$

TRANSFORMING THE DIRAC EQUATION

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \qquad \Rightarrow i\gamma^\mu \partial'_\mu \psi' - m\psi' = 0$$

$$i\gamma^\mu \partial'_\mu (S\psi) - m(S\psi) = 0$$

$$i\gamma^\mu \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu (S\psi) - m(S\psi) = 0$$

S is constant in space time, so we can move it to the left of the derivatives

$$i\gamma^\mu S \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu \psi - mS\psi = 0$$

Now slap S^{-1} from the left

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad S^{-1} \quad \rightarrow \quad i\gamma^\mu S \frac{\partial x^\nu}{\partial x^{\mu'}} \partial_\nu \psi - mS\psi = 0$$

Since these equations must be the same, S must satisfy

$$\gamma^\nu = S^{-1}\gamma^\mu S \ \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

PARITY OPERATOR

- For the parity operator, invert the spatial coordinates while keeping the time coordinate unchanged:

$$P = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right)$$

$$\frac{\partial x_0}{\partial x_0{}'} = 1 \qquad \frac{\partial x_1}{\partial x_1{}'} = -1$$

$$\frac{\partial x_2}{\partial x_2{}'} = -1 \qquad \frac{\partial x_3}{\partial x_3{}'} = -1$$

- We then have

$$\gamma^0 = S^{-1}\gamma^0 S$$

$$\gamma^1 = -S^{-1}\gamma^1 S$$

$$\gamma^2 = -S^{-1}\gamma^2 S$$

$$\gamma^3 = -S^{-1}\gamma^3 S$$

$$\gamma^\nu = S^{-1}\gamma^\mu S \, \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

$$\gamma^0 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

$$(\gamma^0)^2 = 1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0$$

We find that γ^0 satisfies our needs

$$\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

$$\gamma^i = -\gamma^0\gamma^i\gamma^0 = \gamma^0\gamma^0\gamma^i = \gamma^i$$

$$S_P = \gamma^0$$

NEXT TIME

- Read 4.6-4.9 and Chapter 5
- Lots of notation, lots of stuff going on . . . .
  - please stop by if you have questions!