H. A. TANAKA: PHY 489/1489

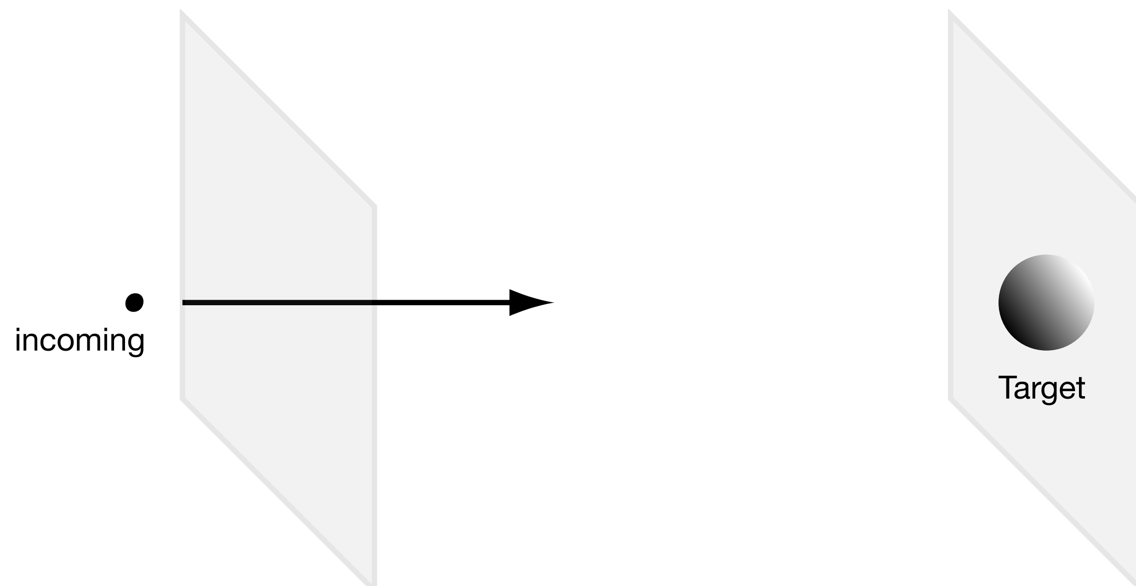
LECTURE 4: PHASE SPACE IN DECAYS

LAST TIME:

- We reviewed some basic quantum mechanics
  - hamiltonian, time evolution
  - angular momentum, Pauli matrices
  - Transition rates, Born approximation, golden rule
  - Decay rates
- Today:
  - phase space
  - decay rate formula
  - (scattering rate)
  - sorry, no pizza

SCATTERING RATES

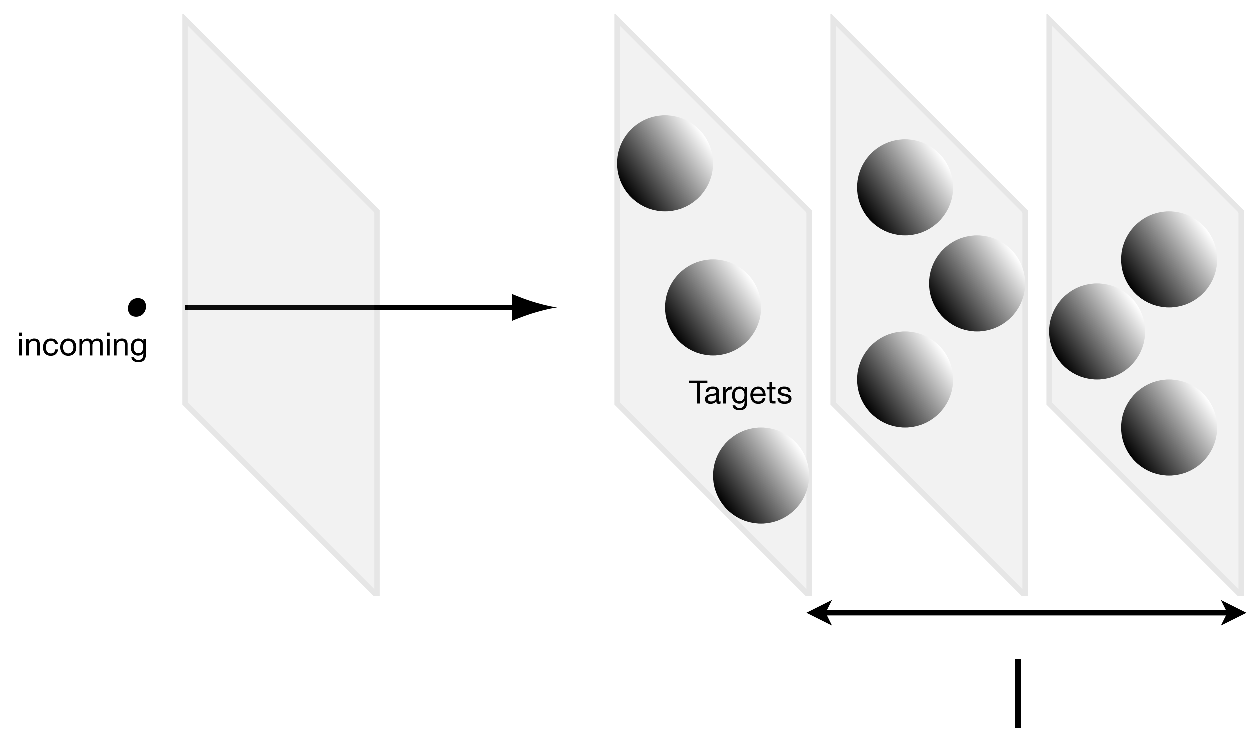
- Send in particles on a "target" and study what comes out
  - if particles are "hard spheres",projectile is infinitesimal



- Probability of interaction: area of target/unit area:
  - area of target particle = "cross section" σ
- Rate $\propto$ rate of incoming particles:
  - Flux ϕ = particles/unit area/time ~ n_i v

MORE THAN ONE TARGET

- More than one "layer" of target particles
- More than one target per unit area.



- Rate $\propto$ targets in the column swept by the incoming beam
  - Rate = N_T/Unit Area x σ x ϕ = n l σ ϕ
  - n = number density of target particles, l = length of target
  - Rate/volume = n σ ϕ

COLLIDING BEAMS:

- In this case, the relative velocity between the two particles determines the flux:
  - $\phi = n_a(v_a+v_b)$
  - Rate $= n_a\,(v_a+v_b)\,n_b\,\sigma$

DIFFERENTIAL CROSS SECTION

- In hard sphere scattering, something "happening" is binary:
  - If the balls hit each other, then something happened
  - otherwise, nothing happened
- Generalize: considering "differential cross section."
  - Probability that particle ends up in a particular part of phase space
  - e.g.. a particular momentum/angle range.

$$\sigma \Rightarrow \frac{d^3\sigma}{d\Omega\ dp}$$

polar angle

$$d\Omega = \sin\theta d\theta\ d\phi = d\cos\theta\ d\phi$$

"solid angle"

azimuthal angle

- Notation lends itself to "integrating" over a phase space variable: say we don't care about the momentum but only the angle:

$$\frac{d\sigma}{d\Omega} = \int p^2 dp \frac{d^3\sigma}{d\Omega\ dp}$$

TOTAL CROSS SECTION

- "total cross section"
  - integrate over all phase space

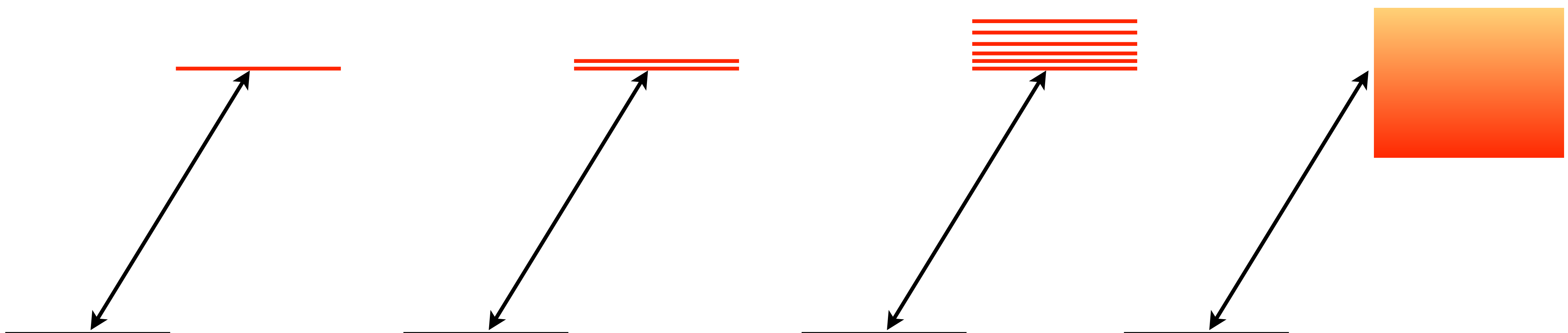
$$\sigma_{TOT} = \int p^2 dp\, d\phi\, d\cos\theta \frac{d^3\sigma}{d\Omega\, dp}$$

  - cross section for a particle to end up anywhere
- For "infinite range" interactions, the total cross section can be infinite; i.e. "something" always happens
  - Reflects the fact that no matter how far you are away, there is still some electric field that will deflect your particle.

GOLDEN RULE

- Fermi's golden rule states that the probability of a transition in quantum mechanics is given by the product of:
  - The absolute value of the matrix element (a k a amplitude) squared
  - The available density of states.

$$P \propto |\mathcal{M}|^2 \times \rho$$



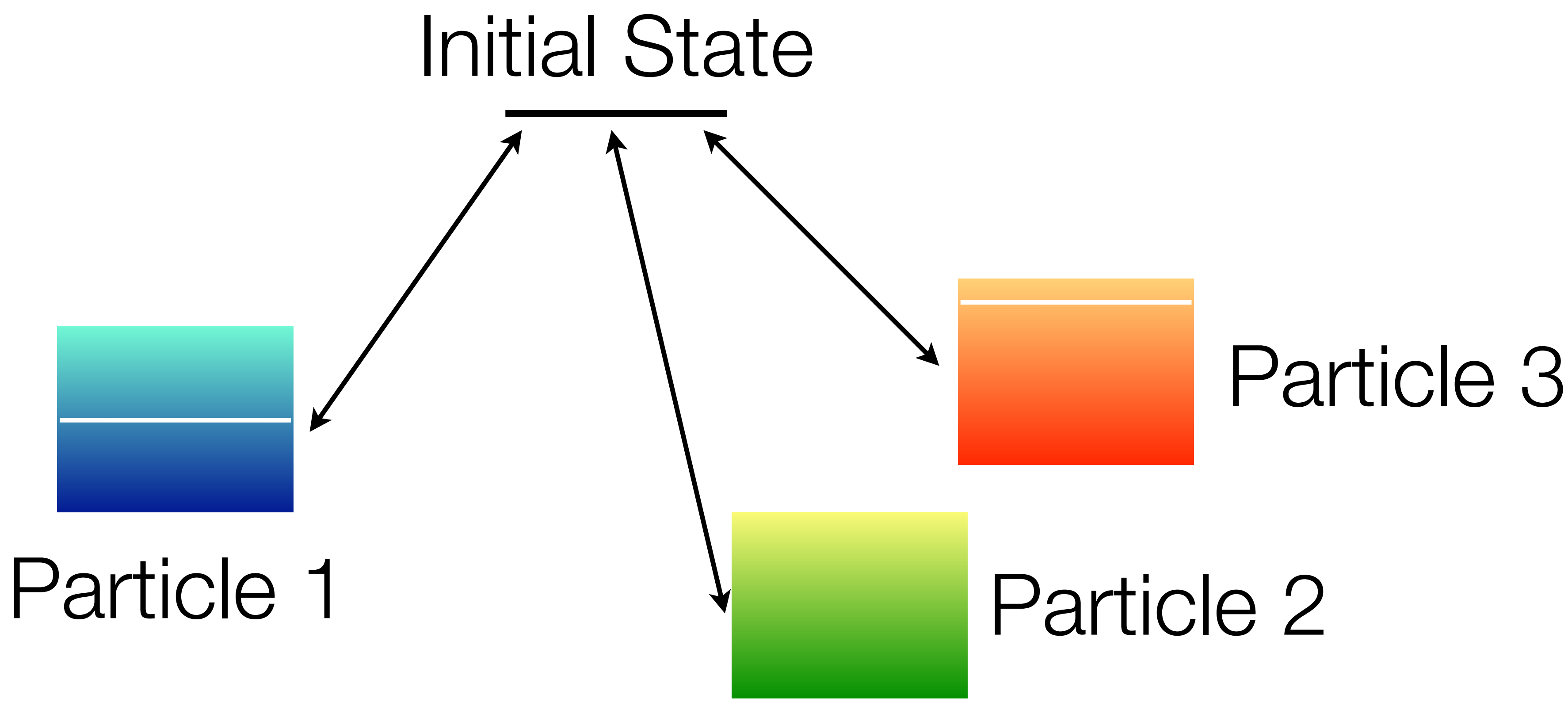
$$P \propto |\mathcal{M}|^2$$

$$P \propto 2 \times |\mathcal{M}|^2$$

$$P \propto \int |\mathcal{M}(E)|^2 \, \rho(E) \, dE$$

- Typically a decay of a particle into states with lighter product masses has more "phase space" and more likely to occur.
- Let's see how to calculate the phase space
  - we'll learn how to calculate amplitudes later.

PRODUCT OF PHASE SPACE



- What is net phase space for the particle 1,2,3 to end up in particular places?
  - 0 if energy and momentum are not conserved
  - 0 if particles are not on "mass shell"
- Otherwise, the product of the individual phase spaces:

$$\rho = \rho_1(p_1^\mu) \times \rho_2(p_2^\mu) \times \rho_3(p_3^\mu)$$

each component of the four-momentum is independent

integral extends over region satisfying kinematic constraints

$$\rho_{tot} = \int_{allowed} \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\ \rho_1(p_1^\mu)\rho_2(p_2^\mu)\rho_3(p_3^\mu)$$

PHASE SPACE IN DECAYS

Symmetry factor

Product over all outgoing particles

Energy must be positive

$$\Gamma = \frac{S}{2m} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu - \sum_f p_f^\mu) \times \prod_{j=f}^{N} 2\pi\ \delta(p_f^2 - m_f^2)\ \Theta(p_f^0)\ \frac{d^4 p_f}{(2\pi)^4}$$

Lorentz Invariant Matrix element factor (function of kinematics, polarizations, etc.)

Energy and momentum must be conserved

Outgoing particles must be on mass shell

distributed evenly in phase space

- Complicating looking, but represents a basic statement:
  - apart from matrix element, phase space is distributed evenly among all particles subject to mass requirements, E/p conservation
  - "dynamics" like parity violation, etc. incorporated into matrix element.

THE SYMMETRY FACTOR:

- Consider the integration of phase space for two particles in the final state where the particles are of the same species.

$$\int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4}$$

- At some point, say, there will be a configuration where $p_1 = K_1$ and $p_2 = K_2$
  - Since the particles are identical, we should also have the reverse case:
    - $p_1 = K_2$, $p_2 = K_1$
    - the integral will contain both cases separately.
  - However, in quantum mechanics, the identicalness of particles of the same species means that these are the same state and we have double counted.
  - We need to add a factor of 1/2 to the phase space
- Likewise, for n identical particles in the final state, we need a factor of 1/n!

PHASE SPACE: 2-BODY DECAY

$$\Gamma \;=\; \frac{S}{2m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu - p_2^\mu - p_3^\mu)$$

$$\times (2\pi)\ \delta(p_2^2 - m_2^2)\ \Theta(p_2^0) \times (2\pi)\ \delta(p_3^2 - m_3^2)\ \Theta(p_3^0)$$

$$\frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4}$$

Let's integrate over overall outgoing particle phase space to get the total decay rate

$$d^4 p \equiv dp^0 dp^1 dp^2 dp^3$$

- Start with the phase space factors:

$$\delta(p^2 - m^2 c^2) = \delta((p^0)^2 - \vec{p}^2 - m^2 c^2)$$

$$\delta(p^2 - m^2 c^2) = \frac{1}{2p^0}\left[\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2 c^2}) + \delta(p^0 + \sqrt{\mathbf{p}^2 + m^2 c^2})\right]$$

- Ignore the 2nd δ function since Θ(p_0) will be 0 whenever p_0 is negative

$$\delta(p^2 - m^2 c^2) = \frac{1}{2p^0}\delta(p^0 - \sqrt{\mathbf{p}^2 + m^2 c^2})$$

$$p_0 \Rightarrow \sqrt{\vec{p}^{\,2} + m^2 c^2}$$

ENERGY/MOMENTUM CONSERVATION

- Now integrate over $p^0{}_3$ and $p^0{}_2$ using the previous relations

$$\Gamma \quad = \quad \frac{S}{2m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu - p_2^\mu - p_3^\mu)$$

$$\times \underline{(2\pi)\ \delta(p_2^2 - m_2^2)\ \Theta(p_2^0)} \times \underline{(2\pi)\ \delta(p_3^2 - m_3^2)}\ \Theta(p_3^0)$$

$$\frac{\cancel{d^4 p_2}}{\cancel{(2\pi)^4}} \frac{\cancel{d^4 p_3}}{\cancel{(2\pi)^4}}$$

$$\frac{d^3\mathbf{p}_2}{(2\pi)^3} \frac{d^3\mathbf{p}_3}{(2\pi)^3} \quad \frac{1}{2 \times \sqrt{\vec{p_2}^{\ 2} + m_2^2 c^2}} \qquad \frac{1}{2 \times \sqrt{\vec{p_3}^{\ 2} + m_3^2 c^2}}$$

note $p^0{}_2$ and $p^0{}_3$ are now set according to E/p conservation by the δ function

$$\Gamma = \frac{S}{32\pi^2 m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\mathbf{p}_2^2 + m_2^2}\sqrt{\mathbf{p}_3^2 + m_3^2}} d^3\mathbf{p}_2 d^3\mathbf{p}_3$$

DECAY AT REST:

- Decompose the product delta function (particle 1 at rest)

$$\delta^4(p_1^\mu - p_2^\mu - p_3^\mu) = \delta\left(m_1 - \sqrt{\mathbf{p}_2^2 + m_2^2} - \sqrt{\mathbf{p}_3^2 + m_3^2}\right)\ \delta^3(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)$$

- Perform the d^3p_3 integral

$$\Gamma = \frac{S}{32\pi^2 m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\mathbf{p}_2^2 + m_2^2}\underbrace{\sqrt{\mathbf{p}_3^2 + m_3^2}}_{\downarrow\ \sqrt{\mathbf{p}_2^2 + m_3^2}}} d^3\mathbf{p}_2 \cancel{d^3\mathbf{p}_3}$$

$$\Gamma = \frac{S}{32\pi^2 m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\mathbf{p}_2^2 + m_2^2}\sqrt{\mathbf{p}_2^2 + m_3^2}} d^3\mathbf{p}_2$$

INTEGRAL IN SPHERICAL COORDINATES

$$d^3\mathbf{p}_2 \Rightarrow d\phi\, d\cos\theta\, |\mathbf{p}_2|^2 d|\mathbf{p}_2|$$

Assume no dependence of M on p

$$\Gamma = \frac{S}{32\pi^2\hbar m_1} \times \int d\phi\, d\cos\theta\, |\mathbf{p}_2|^2\, d|\mathbf{p}_2||\mathcal{M}|^2 \times \frac{\delta(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2c^2})}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\,\sqrt{\mathbf{p}_2^2 + m_3^2c^2}}$$

$$\int_0^{2\pi} d\phi \rightarrow 2\pi \qquad \int_{-1}^{+1} d\cos\theta \rightarrow 2$$

Problem 3.10

$$\begin{aligned} u &= \sqrt{\mathbf{p}_2^2 + m_2^2c^2} + \sqrt{\mathbf{p}_2^2 + m_3^2c^2} \\ du &= \frac{u|\mathbf{p}_2|}{\sqrt{\mathbf{p}_2^2+m_2^2c^2}\sqrt{\mathbf{p}_2^2+m_3^2c^2}} d|\mathbf{p}_2| \end{aligned}$$

$$\Gamma = \frac{S}{8\pi m_1} \times \int du|\mathcal{M}|^2 \times \delta(m_1 - u)\frac{|\mathbf{p}_2|}{u}$$

The final integral over u sends u=m and makes p_2 consistent with E conservation

FINAL RESULT: TWO-BODY DECAY RATE:

$$\Gamma = \frac{S|\mathbf{p}_2|}{8\pi m_1^2}|\mathcal{M}|^2$$

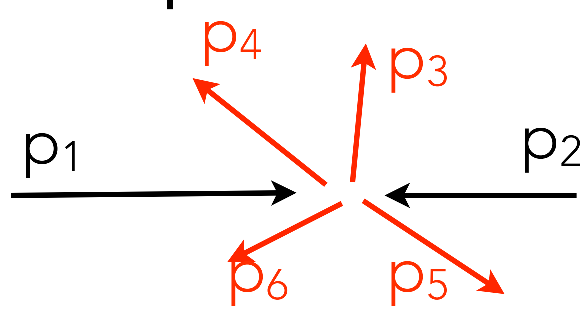
why $|\mathbf{p}_2|$ and not $|\mathbf{p}_3|$?

- Now need to calculate the matrix element $\mathscr{M}$
  - We'll use Feynman diagrams and the associated calculus to calculate amplitudes for various elementary processes

SCATTERING

- Phase space expression for scattering of two particles

p_1 and p_2 are 4-vectors!



$$\sigma \quad = \quad \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \textstyle\sum_f p_f^\mu)$$

$$\times \quad \textstyle\prod_{j=3}^{N} \; 2\pi \; \delta(p_j^2 - m_j^2) \; \Theta(p_j^0) \; \frac{d^4 p_j}{(2\pi)^4}$$

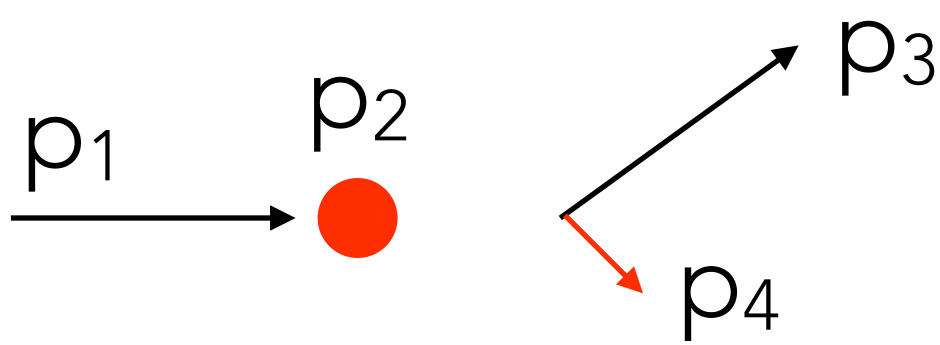
- It has almost the same form as the decay phase space

$$\Gamma \quad = \quad \frac{S}{2m} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu - \textstyle\sum_f p_f^\mu)$$

$$\times \quad \textstyle\prod_{j=2}^{N} \; 2\pi \; \delta(p_j^2 - m_j^2) \; \Theta(p_j^0) \; \frac{d^4 p_j}{(2\pi)^4}$$

LAB-FRAME SCATTERING

- consider e + p → e + p, initial proton at rest



Since $m_e \ll m_p$,

assume $m_e \sim 0$

$$\sigma \quad = \quad \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \textstyle\sum_f p_f^\mu)$$

$$\times \prod_{j=3}^{N} \; 2\pi \; \delta(p_j^2 - m_j^2) \; \Theta(p_j^0) \; \frac{d^4 p_j}{(2\pi)^4}$$

$$\downarrow$$

$$\sigma \quad = \quad \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \textstyle\sum_f p_f^\mu)$$

$$\times \quad \prod_{j=3}^{N} \; \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3}$$

SUMMARY

- Please read 4.1-4.5 for next time