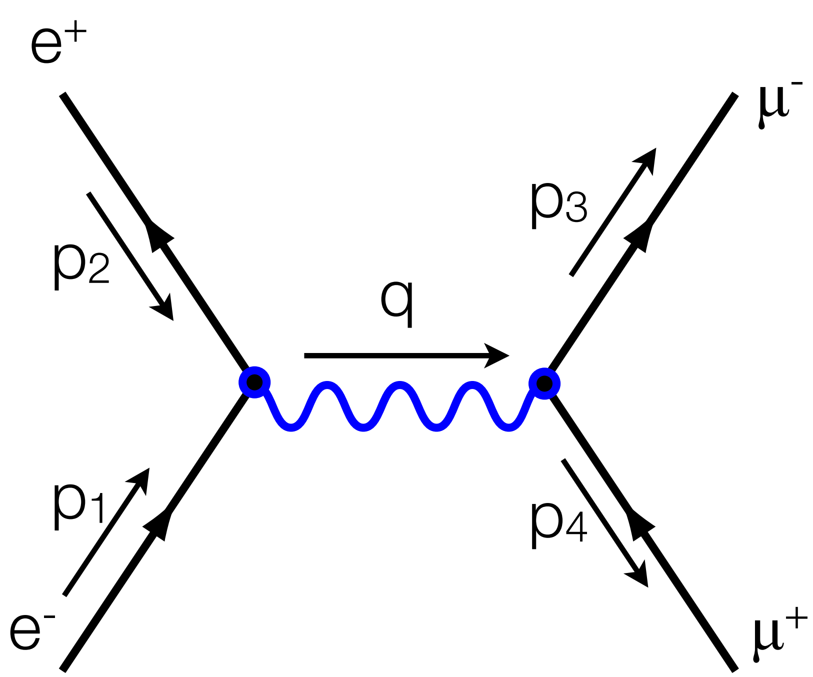
LECTURE 8:

ELECTRON-POSITRON ANNIHILATION

Step I/II: The Feynman Diagram and rules



$$\frac{1}{(2\pi)^4} \int d^4q \;\; \frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}(3)\; ig_e\gamma^\mu \; v(4) \quad (2\pi)^4\delta^4(q - p_3 - p_4)$$

$$\bar{v}(2)\; ig_e\gamma^\nu \; u(1) \quad (2\pi)^4\delta^4(p_1 + p_2 - q)$$

$$[\bar{u}(3)\; \gamma^\mu \; v(4)] \;\; g_{\mu\nu} \;\; [\bar{v}(2)\; \gamma^\nu \; u(1)]$$

$$i(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) \times \frac{g_e^2}{(p_1 + p_2)^2}$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1 + p_2)^2}\, [\bar{u}(3)\; \gamma^\mu \; v(4)]\, [\bar{v}(2)\; \gamma_\mu \; u(1)]$$

WHAT TO DO WITH THIS?

- We need to evaluate the amplitude
- Recall the cross section expression:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{p_f}|}{|\mathbf{p_i}|}|\mathcal{M}|^2$$

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu) \times \prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2 c^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3}$$

- We need to calculate $|M|^2$ and put it into the phase space expression
- What expressions should we use for the spinors?
    - Recall that a Dirac particle basically has three properties
        - particle or antiparticle
        - spin (1/2)
        - momentum

HELICITY STATES

- We had previously constructed helicity states of a Dirac particle along the z-axis

Use “positive” energy solutions

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ p_z c/(E+mc^2) \\ (p_x + ip_y)c/(E+mc^2) \end{pmatrix} \qquad u_2 = N \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \end{pmatrix}$$

electrons

$$-v_2 \equiv u_3 = N \begin{pmatrix} p_z c/(E+mc^2) \\ (p_x + ip_y)c/(E+mc^2) \\ 1 \\ 0 \end{pmatrix} \qquad v_1 \equiv u_4 = N \begin{pmatrix} (p_x - ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \\ 0 \\ 1 \end{pmatrix}$$

Use “negative” energy solutions

positrons

- We now generalize this to any direction

HELICITY OPERATOR:

- Recall that helicity is the projection of the spin onto the direction of motion
- In considering the angular momentum properties, we introduced the spin operator:

$$\mathbf{S} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

- So if we want to project this along the direction of the momentum, we have

$$\mathbf{h} = \frac{1}{|\mathbf{p}|}\mathbf{S}\cdot\mathbf{p} = \frac{\hbar}{2|\mathbf{p}|}\begin{pmatrix} \vec{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \vec{\sigma}\cdot\mathbf{p} \end{pmatrix}$$

HELICITY EIGENSTATES

- By applying **h** to a hypothesized spinor, we can derive the eigenstates
- With polar coordinates:
  - θ = polar angle to z axis
  - ϕ = azimuthal angle atan(p_y/p_z)

s = sin θ/2

c = cos θ/2

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ \frac{p}{E+m}\cos\frac{\theta}{2} \\ \frac{p}{E+m}e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \qquad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \\ \frac{p}{E+m}\sin\frac{\theta}{2} \\ -\frac{p}{E+m}e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}s \\ -\frac{p}{E+m}e^{i\phi}c \\ -s \\ ce^{i\phi} \end{pmatrix} \qquad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}c \\ \frac{p}{E+m}e^{i\phi}s \\ c \\ se^{i\phi} \end{pmatrix}$$

REALTIVISTIC LIMIT

- If E >> m . . . .

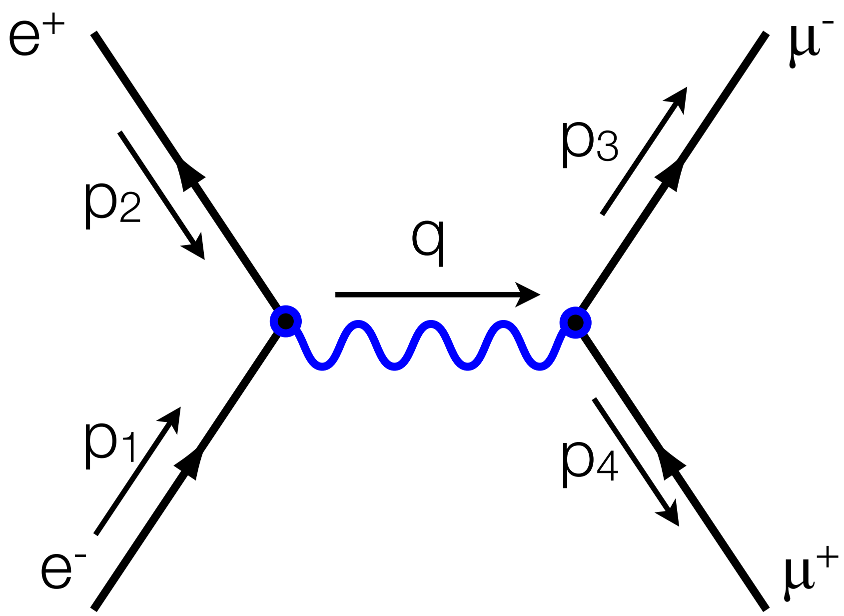
$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}e^{i\phi}s \end{pmatrix} \longrightarrow u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ e^{i\phi}s \end{pmatrix}$$

$$u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}e^{i\phi}c \end{pmatrix} \longrightarrow u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -e^{i\phi}c \end{pmatrix}$$

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}s \\ -\frac{p}{E+m}e^{i\phi}c \\ -s \\ ce^{i\phi} \end{pmatrix} \longrightarrow v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ e^{i\phi}c \end{pmatrix}$$

$$v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}c \\ \frac{p}{E+m}e^{i\phi}s \\ c \\ se^{i\phi} \end{pmatrix} \longrightarrow v_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ e^{i\phi}s \end{pmatrix}$$

INCOMING SPINORS



$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\;\gamma^\mu\;v(4)]\,[\bar{v}(2)\;\gamma_\mu\;u(1)]$$

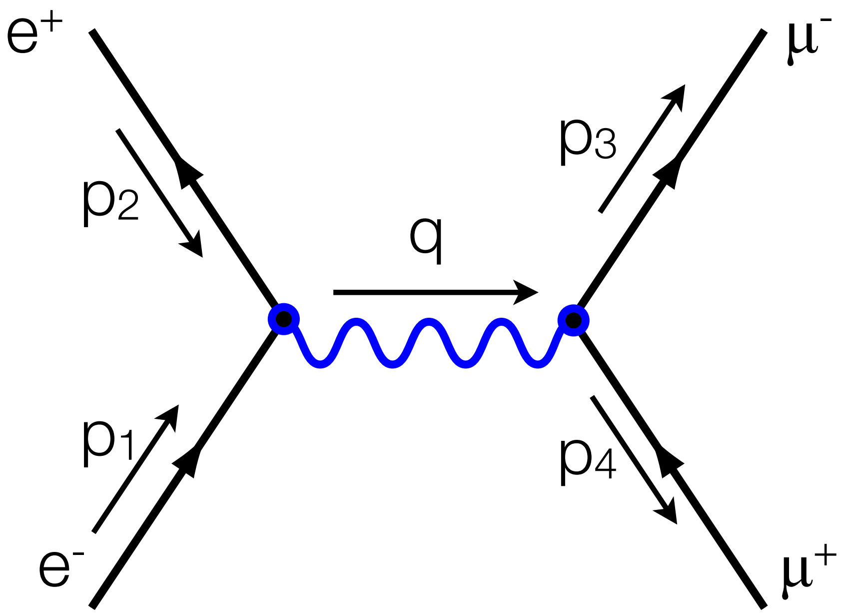
- Initial state: put it along the z-axis
  - incoming electron (θ=0, ϕ = 0)

$$u_\uparrow(p_1) = \sqrt{E_1}\begin{pmatrix} c_1 \\ s_1 e^{i\phi_1} \\ c_1 \\ e^{i\phi_1} s_1 \end{pmatrix} \Rightarrow \sqrt{E_1}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_\downarrow(p_1) = \sqrt{E_1}\begin{pmatrix} -s \\ c_1 e^{i\phi_1} \\ s_1 \\ -c^{i\phi_1} s_1 \end{pmatrix} \Rightarrow \sqrt{E_1}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

- incoming positron (θ=π, ϕ = 0)

$$v_\uparrow(p_2) = \sqrt{E_2}\begin{pmatrix} s_2 \\ -c e^{i\phi_2} \\ -s_2 \\ c_2 e^{i\phi_2} \end{pmatrix} \Rightarrow \sqrt{E_2}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \qquad v_\downarrow(p_2) = \sqrt{E_2}\begin{pmatrix} c_2 \\ s_2 e^{i\phi_2} \\ c_2 \\ e^{i\phi_2} s_2 \end{pmatrix} \Rightarrow \sqrt{E_2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

OUTGOING SPINORS



$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\;\gamma^\mu\;v(4)]\,[\bar{v}(2)\;\gamma_\mu\;u(1)]$$

- Outgoing state:
  - incoming electron (θ_3, 0)

$$u_\uparrow(p_3) = \sqrt{E_3}\begin{pmatrix} c_3 \\ s_3 e^{i\phi_3} \\ c_3 \\ s_3 e^{i\phi_3} \end{pmatrix} \Rightarrow \sqrt{E_3}\begin{pmatrix} c_3 \\ s_3 \\ c_3 \\ s_3 \end{pmatrix} \qquad u_\downarrow(p_3) = \sqrt{E_3}\begin{pmatrix} -s_3 \\ c_3 e^{i\phi_3} \\ s_3 \\ -c_3 e^{i\phi_3} \end{pmatrix} \Rightarrow \sqrt{E_3}\begin{pmatrix} -s_3 \\ c_3 \\ s_3 \\ -c_3 \end{pmatrix}$$

- incoming positron (θ_4=π-θ_3, ϕ = π)

$$v_\uparrow(p_4) = \sqrt{E_4}\begin{pmatrix} s_4 \\ -c_4 e^{i\phi_4} \\ -s_4 \\ c_4 e^{i\phi_4} \end{pmatrix} \Rightarrow \sqrt{E_4}\begin{pmatrix} c_3 \\ s_3 \\ -c_3 \\ -s_3 \end{pmatrix} \qquad v_\downarrow(p_4) = \sqrt{E_4}\begin{pmatrix} c_4 \\ s_4 e^{i\phi_4} \\ c_4 \\ s_4 e^{i\phi_4} \end{pmatrix} \Rightarrow \sqrt{E_4}\begin{pmatrix} s_3 \\ -c_3 \\ s_3 \\ -c_3 \end{pmatrix}$$

HELICITY COMBINATIONS

- Now we can consider any combinations of felicities by placing the appropriate spinors in the expression

$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,\underbrace{[\bar{u}(3)\ \gamma^\mu\ v(4)]}_{}\,[\bar{v}(2)\ \gamma_\mu\ u(1)]$$

j_μ j_e

- Note that in general we will consider products like

$$\bar{\psi}\gamma^\mu\phi = \psi^\dagger\gamma^0\gamma^\mu\phi$$

$$\bar{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi \quad = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4$$

$$\bar{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi \quad = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$

$$\bar{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi \quad = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1)$$

$$\bar{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi \quad = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

j_μ

- We can try the "↑↓" or "RL" combination

$$u_R = \sqrt{E}\begin{pmatrix} c_3 \\ s_3 \\ c_3 \\ s_3 \end{pmatrix} \qquad v_L = \sqrt{E}\begin{pmatrix} s_3 \\ -c_3 \\ s_3 \\ -c_3 \end{pmatrix}$$

$$\bar{u}_R \gamma^\mu v_L$$

$$\bar{u}_R \gamma^0 v_L = E \times (cs - cs + cs - cs) = 0$$
$$= \psi_1^* \phi_1 + \psi_2^* \phi_2 + \psi_3^* \phi_3 + \psi_4^* \phi_4$$

$$\bar{u}_R \gamma^1 v_L = E \times (-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta$$
$$= \psi_1^* \phi_4 + \psi_2^* \phi_3 + \psi_3^* \phi_2 + \psi_4^* \phi_1$$

$$\bar{u}_R \gamma^2 v_L = -iE \times (-c^2 - s^2 - c^2 - s^2) = 2ie(c^2 + s^2) = 2iE$$
$$= -i(\psi_1^* \phi_4 - \psi_2^* \phi_3 + \psi_3^* \phi_2 - \psi_4^* \phi_1)$$

$$\bar{u}_R \gamma^3 v_L = E \times (cs + sc + cs + sc = 4Esc = 2E\sin\theta$$
$$= \psi_1^* \phi_3 - \psi_2^* \phi_4 + \psi_3^* \phi_1 - \psi_4^* \phi_2$$

$$\bar{u}_L \gamma^\mu v_R = 2E(0, -\cos\theta, -i, \sin\theta)$$
$$\bar{u}_R \gamma^\mu v_R = 2E(0,0,0,0)$$

$$\bar{u}_R \gamma^\mu v_L = 2E(0, -\cos\theta, i, \sin\theta)$$

$$\bar{u}_L \gamma^\mu v_L = 2E(0,0,0,0)$$

j_e

- By the same methods, can show:

$$\bar{u}_L\gamma^\mu v_L = 2E(0,0,0,0) \qquad \bar{u}_R\gamma^\mu v_R = 2E(0,0,0,0)$$

$$\bar{u}_L\gamma^\mu v_R = 2E(0,-1,i,0) \qquad \bar{u}_R\gamma^\mu v_L = 2E(0,-1,-i,0)$$

- and we can combine with jm to get the amplitude for any particular helicity combination

$$\bar{u}_R\gamma^\mu v_L = 2E(0,-\cos\theta,i,\sin\theta) \qquad \bar{u}_L\gamma^\mu v_R = 2E(0,-\cos\theta,-i,\sin\theta)$$

$$\begin{aligned}\mathcal{M}_{LR\to LR} &= -\frac{e^2}{4E^2}[\bar{u}_{3L}\gamma^\mu v_{4R}]\,[\bar{v}_{2R}\gamma_\mu u_{1L}] \\ &= -\frac{4e^2E^2}{(p_1+p_2)^2}(0-\cos\theta-1,0) \\ &= e^2(1+\cos\theta) = \mathcal{M}_{RL\to RL}\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{LR\to RL} &= -\frac{e^2}{4E^2}[\bar{u}_{3R}\gamma^\mu v_{4L}]\,[\bar{v}_{2R}\gamma_\mu u_{1L}] \\ &= -e^2\times(0-\cos\theta+1+0) \\ &= e^2\times(-\cos\theta+1) \\ &= \mathcal{M}_{RL\to LR}\end{aligned}$$

ENDGAME

- Put these into our differential cross section:
  - and recall that s = $(p_1+p_2)^2=(2E)^2$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{p_f}|}{|\mathbf{p_i}|}|\mathcal{M}|^2$$

$$|\mathcal{M}_{LR\to LR}|^2 = |\mathcal{M}_{RL\to RL}|^2 = e^4(1+\cos\theta)^2$$

$$|\mathcal{M}_{LR\to RL}|^2 = |\mathcal{M}_{RL\to LR}|^2 = e^4(1-\cos\theta)^2$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{256\pi^2 E^2}(1\pm\cos\theta)^2$$

- The unpolarized spin-averaged cross section
  - if we detect all outgoing helicity states, add up all the contributions
  - if the beams are unpolarized (i.e. initial states are half of each helicity), each helicity combination is 1/4 of the total beam, so we divide by 4

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}(1+\cos^2\theta)$$

NEXT TIME

- Please read rest of Chapter 6