PHYSICS 489/1489

LECTURE 7:
QUANTUM ELECTRODYNAMICS

REMINDER

- Problem set 1 due today
  - 1700 in Box #7
- If you did not get an email from me on Monday night regarding the problem set, please email me.
  - I am using emails in blackboard
  - If that is not where you get your email, please let me know.

THE PHOTON

- Apart from $\bar{\psi}\psi$ we need some other particle/object with definite Lorentz transformation properties to make Lorentz invariants
  - What would we do with the "vector" term $\bar{\psi}\gamma^{\mu}\psi$ to get a Lorentz scalar?
- Recall the photon:
  - Classically, we have Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{J}$$

  - Recall that we can re-express the Maxwell equations using potentials:

$$\mathbf{E} = -\nabla\phi \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

  - these can in turn be combined to make a 4 vector:
  - Likewise for the "source" terms ρ **and J:**

$$A^{\mu} = (\phi, \mathbf{A})$$

$$J^{\mu} = (c\rho, \mathbf{J})$$

MAXWELL'S EQUATION

- All four equations can be expressed as:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- The issue is that A is (far) from unique:
    - Consider: $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$

$$\partial_\mu \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu (\partial_\mu (A^\mu + \partial^\mu \lambda) =$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu (A^\mu) + \partial_\mu \partial^\mu \partial^\nu \lambda - \partial^\nu \partial_\mu \partial^\mu \lambda$$

    - last terms cancel, so "new" A_μ is also a solution to Maxwell's equation
    - they are physically the same, so we can make some conventions:
- "Lorentz gauge condition": $\partial_\mu A^\mu = 0$ $\quad \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$
- "Coulomb gauge" $\quad A^0 = 0 \quad \nabla \cdot \mathbf{A} = 0$

"FREE" SOLUTIONS

- "Free" means no sources (charges, currents): $J^{\mu}=0$
- Find solution by ansatz: $$\partial^{\mu}\partial_{\mu}A^{\nu}=0$$

$$A^{\mu}(x)=a\ e^{-ip\cdot x}\epsilon^{\mu}(p)$$

- Now check:

$$\partial_{\mu}A^{\nu}(x)=-ip_{\mu}\ a\ e^{-ip\cdot x}\epsilon^{\nu}(p)$$

$$\partial^{\mu}\partial_{\mu}A^{\nu}(x)=(-i)^2p^{\mu}p_{\mu}\ a\ e^{-ip\cdot x}\epsilon^{\nu}(p)=0$$

$$\partial_{\mu}A^{\mu}=0\Rightarrow p_{\mu}\epsilon^{\mu}(p)=0$$

$$p^2=m^2c^2=0$$

$$A^0=0\Rightarrow\epsilon^0=0$$

$$\Rightarrow\mathbf{p}\cdot\epsilon=0$$

- Conclusions:
  - Photon is massless
  - Polarization ε is transverse to photon direction (in Coulomb gauge):
    - it has two degrees of freedom/polarizations

MAIKING A SCALAR OBJECT

- In the end, these spaces must collapse:
  - In Lorentz space, this happens by contracting indices: $g_{\mu\nu} a^\mu b^\nu = a^\mu b_\mu$
  - In spinor space, products of adjoint spinors with spinors (with gamma matrices possibly in between): $\bar{u_1} \Gamma v_2$ Γ=(product of g matrices)
- but some expressions have structure in both:

sum over μ collapses the Lorentz structure

$$\mathcal{M} = -\frac{g_e^2}{(p_1 + p_2)^2} [\bar{u}(3)\ \gamma^\mu\ v(4)]\ [\bar{v}(2)\ \gamma_\mu\ u(1)]$$

Contracted in spinor space, but not in Lorentz

Same here

REMINDER OF DIRAC SPINORS

- States:

"positive" energy solutions

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ p_z c/(E+mc^2) \\ (p_x + ip_y)c/(E+mc^2) \end{pmatrix} \qquad u_2 = N \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \end{pmatrix}$$

electrons

$$-v_2 \equiv u_3 = N \begin{pmatrix} p_z c/(E+mc^2) \\ (p_x + ip_y)c/(E+mc^2) \\ 1 \\ 0 \end{pmatrix} \qquad v_1 \equiv u_4 = N \begin{pmatrix} (p_x - ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \\ 0 \\ 1 \end{pmatrix}$$

"negative" energy solutions

positrons

A SECOND LOOK AT DIRAC EQUATION

electrons

$$\psi(x) = ae^{-(i/\hbar)p\cdot x}\ u^s(p)$$

positrons

$$\psi(x) = ae^{(i/\hbar)p\cdot x}\ v^s(p)$$

- "s" labels the spin states (two for electrons/positrons)
- exponential term sets the space/time = energy/momentum
- "spinor" u,v which determines the "Dirac structure":
    - If we insert ψ into the Dirac equation, we get:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu - mc)u^s(p) = 0$$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \Rightarrow (-\gamma^\mu p_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu + mc)v^s(p) = 0$$

"momentum space Dirac equations"

- If we take the adjoint of these equations, we get:

$$\bar{u}^s(\gamma^\mu p_\mu - mc) = 0 \qquad \bar{v}^s(\gamma^\mu p_\mu + mc) = 0$$

ORTHOGONALITY/COMPLETENESS

- From the explicit form of our u/v spinors:

$$\bar{u}^i u^j = 2mc\,\delta^{ij} \qquad \bar{v}^i v^j = -2mc\,\delta^{ij} \qquad \bar{u}^i v^j = \bar{v}^i u^j = 0$$

- We can also show:

$$\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\mu p_\mu + mc) \qquad \sum_{s=1,2} v^s \bar{v}^s = (\gamma^\mu p_\mu - mc)$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} (b_1, b_2, b_3, b_4) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \\ a_4 b_1 & a_4 b_2 & a_4 b_3 & a_4 b_4 \end{pmatrix}$$

PHOTONS:POLARIZATION/ORTHOGONALITY

- We showed that the polarization 4-vector ε^{μ} with the Lorentz and Coloumb gauge conditions must satisfy:

$$\mathbf{p} \cdot \epsilon = 0$$

- We noted that this allows two degrees of freedom corresponding to transversely polarized electromagnetic fields.
  - We need to two orthogonal ε basis vectors to span the space
- If the photon is moving in the z direction, we can choose:

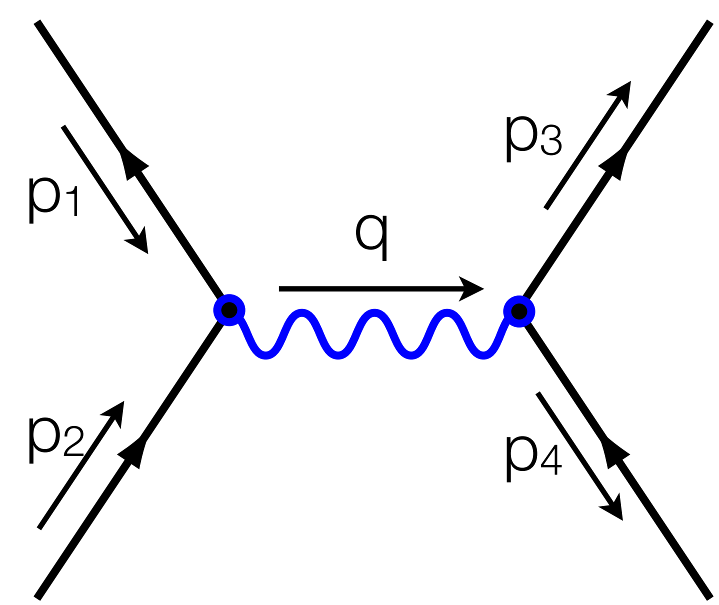
$$\epsilon_{\mu}^{1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_{\mu}^{2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- The polarization vectors satisfy orthogonality/completeness relations:

$$\epsilon_{\mu}^{i*} \epsilon^{\mu j} = -\delta^{ij} \qquad \sum_{s=1,2} \epsilon_i^s \epsilon_j^{s*} = \delta_{ij} - \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j$$

FEYNMAN RULES: EXTERNAL LINES

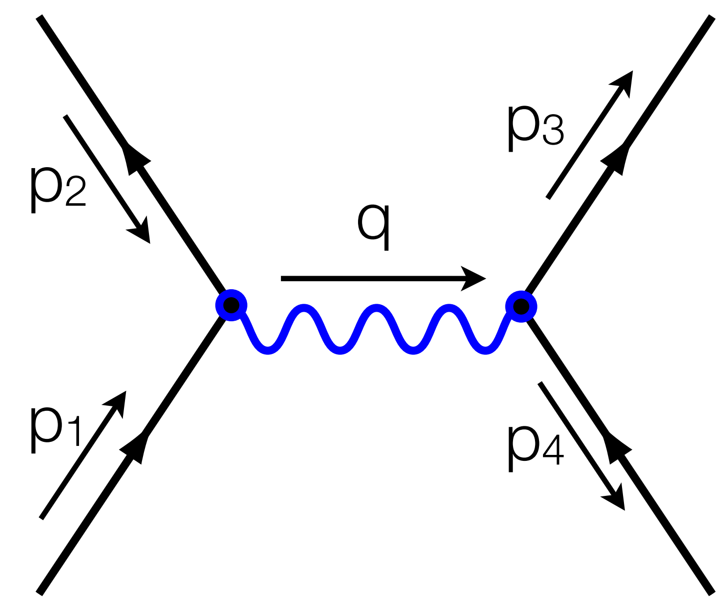
- Right down the Feynman diagram(s) for the process and label the momentum flow
  - use p's for external lines, q's for internal (Griffiths convention).
  - Note that there are two flows:
    - "particle/antiparticle"
    - momentum
    - These are separate



- Now the components of the expression
  - External Lines:
    - Electrons: incoming $u^s(p)$ outgoing $\bar{u}^s(p)$
    - Positrons: incoming $\bar{v}^s(p)$ outgoing $v^s(p)$
    - Photons: incoming $\epsilon_\mu(p)$ outgoing $\epsilon^*_\mu(p)$

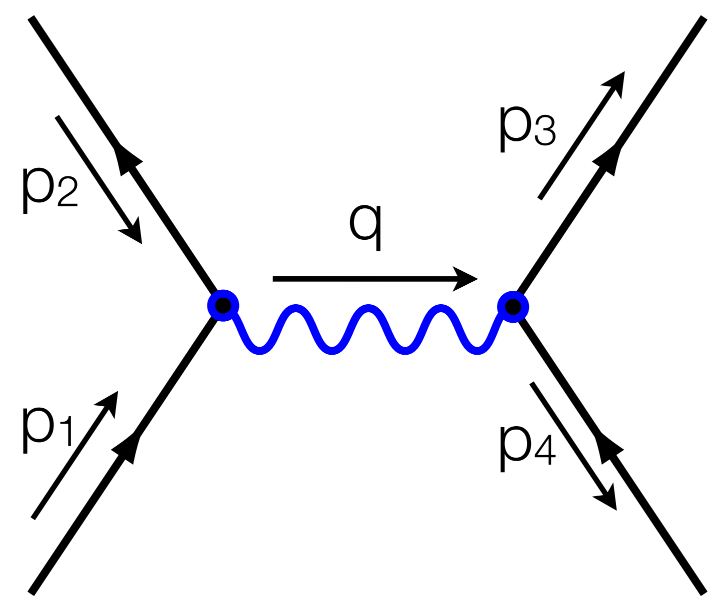
VERTICES AND PROPAGATORS

- For each QED vertex: $ig_e\gamma^\mu \; (2\pi)^4\delta^4(k_1+k_2+k_3)$
  - momentum is "+" incoming, "-" outgoing from vertex
  - g_e is the electromagnetic coupling (Qe)
- Internal lines:
  - electron/positron propagator $\dfrac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$
  - Photon propagator $\dfrac{-ig_{\mu\nu}}{q^2}$
    - indices match vertices/polarization
  - Integral over momentum: $\dfrac{d^4q}{(2\pi)^4}$



- Finally: cancel the overall delta function, what remains is -iM

EXAMPLE

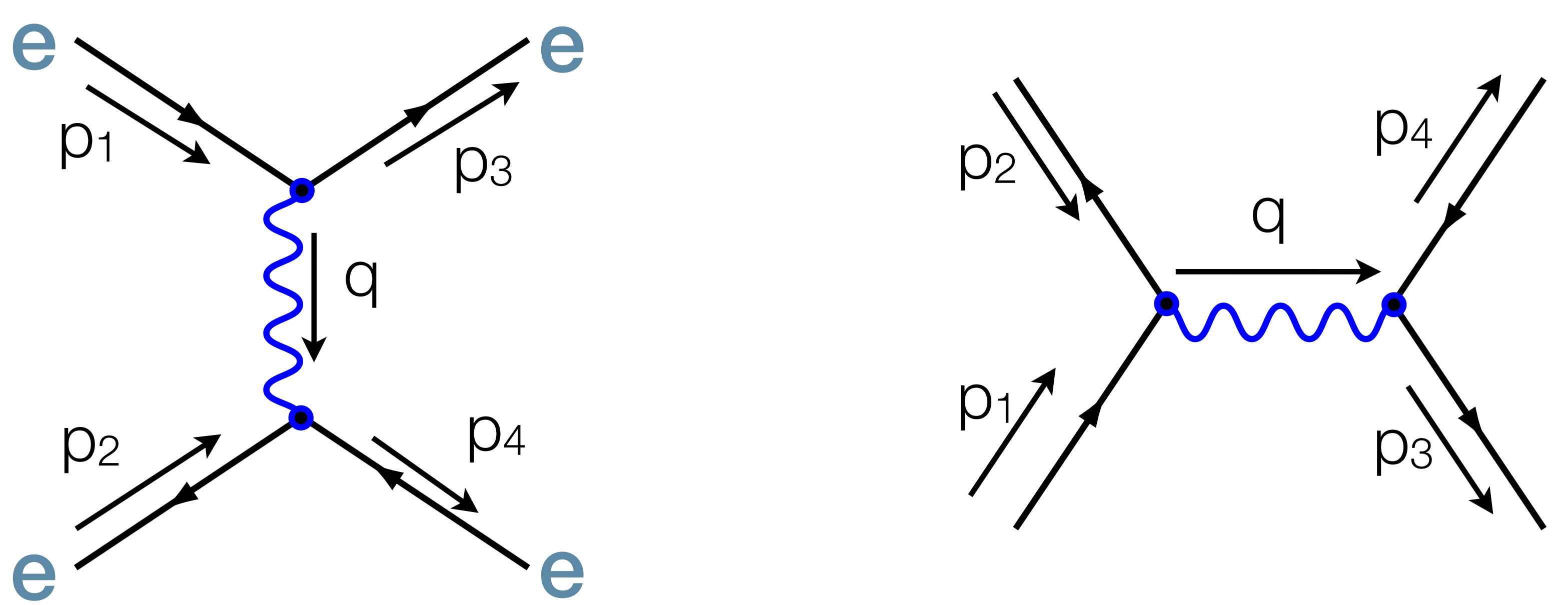


- Order matters due to Dirac matrix structure (photon part doesn't care)
- Griffiths: go backward through the fermion lines:
  - In the "final state": $\bar{u}(3)\ ig_e\gamma^\mu\ v(4) \quad (2\pi)^4\delta^4(q-p_3-p_4)$
  - In the "initial state": $\bar{v}(2)\ ig_e\gamma^\nu\ u(1) \quad (2\pi)^4\delta^4(p_1+p_2-q)$
  - Throw in the internal photon propagator: $\frac{1}{(2\pi)^4}\int d^4q \quad \frac{-ig_{\mu\nu}}{q^2}$

$$i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\frac{g_e^2}{(p_1+p_2)^2}\ [\bar{u}(3)\ \gamma^\mu\ v(4)]\ g_{\mu\nu}\ [\bar{v}(2)\ \gamma^\nu\ u(1)]$$

$$\mathcal{M}=-\frac{g_e^2}{(p_1+p_2)^2}\ [\bar{u}(3)\ \gamma^\mu\ v(4)]\,[\bar{v}(2)\ \gamma_\mu\ u(1)]$$

EXAMPLE: $e^+ + e^- \rightarrow e^+ + e^-$



$$\bar{u}(3)\ ig_e\gamma^\mu\ u(1)\ \ \bar{v}(2)\ ig_e\gamma^\nu\ v(4)\ \ \frac{-ig_{\mu\nu}}{(p_1-p_3)^2}$$

$$\bar{u}(3)\ ig_e\gamma^\rho\ v(4)\ \ \bar{v}(2)\ ig_e\gamma^\sigma\ u(1)\ \ \frac{-ig_{\rho\sigma}}{(p_1+p_2)^2}$$

$$(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$$

NEXT TIME:

- Please read 6.1 and 6.2
  - I would explicitly work out spin summation procedure in 6.2.1 and 6.2.4
- Unfortunately I will be out of town again on Monday
  - If you were planning to come to office hours and cannot come on Tuesday, please contact me and we'll figure something out.
  - If it's urgent, I have some time after class today.