H. A. TANAKA

LECTURE 6:
MORE PROPERTIES OF THE DIRAC EQUATION

TRANSFORMING THE DIRAC EQUATION:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \qquad \Rightarrow \qquad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

$$i\hbar\gamma^\mu\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu(S\psi) - mc\,(S\psi) = 0$$

S is constant in space time, so we can move it to the left of the derivatives

$$i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,(S\psi) = 0$$

Now slap S^{-1} from both sides

$$i\hbar\gamma^\nu\partial_\nu\psi - mc\,\psi = 0 \qquad S^{-1} \to i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,S\psi = 0$$

Since these equations must be the same, S must satisfy

$$\gamma^\nu = S^{-1}\gamma^\mu S\,\frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

EXAMPLE: THE PARITY OPERATOR

- For the parity operator, we want to invert the spatial coordinates while keeping the time coordinate unchanged:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\frac{\partial x_0}{\partial x_0{}'} = 1 \qquad \frac{\partial x_1}{\partial x_1{}'} = -1$$

$$\frac{\partial x_2}{\partial x_2{}'} = -1 \qquad \frac{\partial x_3}{\partial x_3{}'} = -1$$

- We then have

$$\gamma^0 = S^{-1}\gamma^0 S$$
$$\gamma^1 = -S^{-1}\gamma^1 S$$
$$\gamma^2 = -S^{-1}\gamma^2 S$$
$$\gamma^3 = -S^{-1}\gamma^3 S$$

Recalling

$$\gamma^\nu = S^{-1}\gamma^\mu S \, \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\gamma^0)^2 = 1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0$$

We find that γ^0 satisfies our needs

$$\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$
$$\gamma^i = -\gamma^0\gamma^i\gamma^0 = \gamma^0\gamma^0\gamma^i = \gamma^i$$

$$S_P = \gamma^0$$

LORENTZ COVARIANT QUANTITIES

- Recall that for four vectors a^μ, b^μ
  - a^μ, b^μ transform as Lorentz vectors (obviously)
  - $a^\mu b_\mu$ is a scalar (does not change under Lorentz transformations
  - $a^\mu b^\nu$ is a tensor (each has a Lorentz transformation)
- From the previous discussion, we know:
  - Dirac spinors have four components, but don’t transform as Lorentz vectors
  - How do combinations of Dirac spinors change under Lorentz Transformations?

HOW DO WE CONSTRUCT A SCALAR?

- We can use γ^0: define: $\bar{\psi} = \psi^\dagger \gamma^0$
  - Consider a Lorentz transformation with S acting on the spinor
  - We can also show generally that $S^\dagger \gamma^0 S = \gamma^0$
  - This gives us $\bar{\psi}\psi \Rightarrow \psi^\dagger S^\dagger \gamma^0 S \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$
  - so this is a Lorentz invariant
- We can construct the parity operator to check how $\bar{\psi}\psi$ transforms under the parity operation.
  - Recall $S_P = \gamma^0$
    - We can investigate how $\bar{\psi}\psi$ transforms under parity

$$\bar{\psi}\psi \Rightarrow (\psi^\dagger S_P^\dagger \gamma^0)(S_P \psi) = \psi^\dagger \gamma^0 \gamma^0 \gamma^0 \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$$

$\bar{\psi}\psi$ doesn't change sign under parity
it is a Lorentz scalar

THE γ^5 OPERATOR

- Define the operator γ^5 as:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- It anticommutes with all the other γ matrices:

$$\{\gamma^\mu, \gamma^5\} = 0$$

  - use the canonical anti-commutation relations to move γ^μ to the other side
  - γ^μ will anti-commute with for $\mu \neq \nu$
  - γ^μ will commute when $\mu = \nu$
- We can then consider the quantity $\bar{\psi}\gamma^5\psi$
  - Can show that this is invariant under Loretnz transformation.
- What about under parity?

$$\bar{\psi}\gamma^5\psi \Rightarrow (\psi^\dagger S_P^\dagger)\gamma^0\gamma^5(S_P\psi) = -(\psi^\dagger S_P^\dagger)\gamma^0 S_P\gamma^5\psi = -\psi^\dagger\gamma^0\gamma^5\psi = -\bar{\psi}\gamma^5\psi$$

$\bar{\psi}\gamma^5\psi$ is a “pseudoscalar”

OTHER COMBINATIONS

- We can use γ^μ to make vectors and tensor quantities:

$\bar{\psi}\psi$	scalar	1 component	
$\bar{\psi}\gamma^5\psi$	pseudoscalar	1 component	
$\bar{\psi}\gamma^\mu\psi$	vector	4 components	
$\bar{\psi}\gamma^\mu\gamma^5\psi$	pseudovector	4 components	
$\bar{\psi}\sigma^{\mu\nu}\psi$	antisymmetric tensor	6 components	$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

- You can tell the transformation properties by looking at the Lorentz indices
  - γ^5 introduces a sign (adds a "pseudo")
  - Every combination of $\psi^*_i\psi_j$ is a linear combination of the above.
- We will see that the above are the basis for creating interactions with Dirac particles. Interactions will be classified as "vector" or "pseudovector", etc.

Angular Momentum and the Dirac Equation:

- Conservation:
  - In quantum mechanics, what is the condition for a quantity to be conserved?

$$[H, Q] = 0$$

  - Free particle Hamiltonian in non-relativistic quantum mechanics:

$$H = \frac{p^2}{2m} \qquad [H, p] = [\frac{p^2}{2m}, p] = \frac{1}{2m}[p^2, p] = 0$$

  - thus we conclude that the momentum p is conserved
- If we introduce a potential:

$$[H, p] = [\frac{p^2}{2m} + V(x), p] = \frac{1}{2m}[p^2 + V(x), p] \neq 0$$

  - thus, momentum is not conserved

Hamiltonian for the Dirac Particle:

- Starting with the Dirac equation,

$$(\gamma^\mu p_\mu - mc)\psi = 0$$

- determine the Hamiltonian by solving for the energy

- Hints:

$$(\gamma^0)^2 = 0$$

- Answer: $H = c\gamma^0 \left(\gamma \cdot \mathbf{p} + mc\right)$

Is orbital angular momentum conserved?

- We want to evaluate $[H, \vec{L}]$
- Recall: $\vec{L} = \vec{x} \times \vec{p}$ $\qquad L_i = \epsilon_{ijk} x^j p^k$

$$H = c\gamma^0 \left(\gamma \cdot \mathbf{p} + mc\right) \qquad H = c\gamma^0 \left(\delta_{ab}\gamma^a p^b + mc\right)$$

$$[H, L_i] = [c\gamma^0 \left(\delta_{ab}\gamma^a p^b + mc\right), \epsilon_{ijk} x^j p^k]$$

- which parts do not commute?
- $$[c\gamma^0 \delta_{ab}\gamma^a p^b, \epsilon_{ijk} x_j p_k] \qquad [mc, \epsilon_{ijk} x_j p_k]$$

$$c\gamma^0 \delta_{ab}\epsilon_{ijk}[p^b, x^j p^k] \qquad [A, BC] = [A, B]C + B[A, C]$$

$$c\delta_{ab}\epsilon_{ijk}\gamma^0\gamma^a(-i\hbar\delta^{bj}p^k) \qquad -i\hbar c\gamma^0\epsilon_{ijk}\gamma^j p^k \qquad -i\hbar c\gamma^0(\vec{\gamma} \times \vec{p})$$

Orbital angular momentum is not conserved

Spin

- Consider the operator $\vec{S} = \frac{\hbar}{2}\vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ acting on Dirac spinors
- Note that it satisfies all the properties of an angular momentum operator.
- Let's consider the commutator of this with the hamiltonian $H = c\gamma^0 \left(\gamma \cdot \mathbf{p} + mc\right)$

$$\frac{\hbar c}{2}[\gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 mc, \vec{\Sigma}]$$

- once again, consider in component/index notation

$$[H, S^i] = \frac{\hbar c}{2}[\gamma^0 \delta_{ab}\gamma^a p^b + \gamma^0 mc, \Sigma^i]$$

- which part doesn't commute?

$$\frac{\hbar c}{2}\delta_{ab}p^b[\gamma^0\gamma^a, \Sigma^i] \qquad [AB, C] = [A, C]B + A[B, C]$$

$$\frac{\hbar c}{2}\delta_{ab}p^b\gamma^0[\gamma^a, \Sigma^i] \qquad [\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}]$$

$$\begin{pmatrix} 0 & [\sigma_a, \sigma_i] \\ -[\sigma^a, \sigma_i] & 0 \end{pmatrix} \qquad \epsilon_{aij}\begin{pmatrix} 0 & \sigma_j \\ -\sigma^j & 0 \end{pmatrix} \qquad \epsilon_{aij}\gamma^j$$

The "total" spin operator:

- Define the operator:

$$\mathbf{S}^2 = \mathbf{S} \cdot \mathbf{S} \qquad\qquad \mathbf{S} = \frac{\hbar}{2}\boldsymbol{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

- Calculate its eigenvalue for an arbitrary Dirac spinor:

- If the eigenvalue of the operator gives s(s+1), where s is the spin, what is the spin of a Dirac particle?

NEXT TIME

- Please turn in problem set 1 by 1600 on Thursday (6 Oct)
- Please read Chapter 5