H. A. TANAKA, PHYSICS 489/1489

LECTURE 6:
THE DIRAC EQUATION

RELATIVISTIC WAVE EQUATIONS:

- In non-relativistic quantum mechanics, we have the Schrödinger Equation:

$$\mathbf{H}\psi = i\hbar\frac{\partial}{\partial t}\psi \qquad \mathbf{H} = \frac{\mathbf{p}^2}{2m} \qquad \mathbf{p} \Leftrightarrow -i\hbar\nabla$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$$

- Inspired by this, Klein and Gordon (and actually Schrödinger) tried:

$$E^2 = p^2c^2 + m^2c^4 = c^2(-\hbar^2\nabla^2 + m^2c^2)\psi = -\hbar^2\frac{\partial^2}{\partial t^2}\psi$$

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = \frac{m^2c^2}{\hbar^2}\psi$$

$$\partial_\mu = (\partial_0,\ \partial_1,\ \partial_2,\ \partial_3) = \left(\frac{\partial}{\partial ct},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) \qquad (-\hbar^2\partial^\mu\partial_\mu + m^2c^2)\psi = 0$$

"Manifestly Lorentz Invariant"

ISSUES WITH KG AND DIRAC:

- Within the context of quantum mechanics, this had some issues:
  - As it turns out, this allows negative probability densities: $|\psi|^2 < 0$
  - Dirac traced this to the fact that we had second-order time derivative
    - "factor" the E/p relation to get linear relations and obtained:

$$p_\mu p^\mu - m^2c^2 = 0 \Rightarrow (\alpha^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

- and found that:

$$\alpha^\kappa = \gamma^\kappa$$
$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- Dirac found that these relationships could be obtained by matrices, and that the corresponding wave function must be a "vector".

$$\gamma^\mu p_\mu - mc = 0 \Rightarrow (i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

THE DIRAC EQUATION IN ITS MANY FORMS:

$$(i\hbar \not{\partial} - mc)\psi = 0 \qquad \not{a} \equiv a_\mu \gamma^\mu$$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \qquad \not{a} \equiv a_\mu \gamma^\mu = a_0\gamma^0 - a_1\gamma^1 - a_2\gamma^2 - a_3\gamma^3$$

$$\partial_\mu = (\partial_0,\ \partial_1,\ \partial_2,\ \partial_3) = \left(\frac{\partial}{\partial ct},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)$$

$$\left[i\hbar(\gamma^0\partial_0 - \gamma^1\partial_1 - \gamma^2\partial_2 - \gamma^3\partial_3) - mc\right]\psi = 0$$

$$\left[i\hbar\left(\gamma^0\frac{\partial}{\partial ct} - \gamma^1\frac{\partial}{\partial x} - \gamma^2\frac{\partial}{\partial y} - \gamma^3\frac{\partial}{\partial z}\right) - mc\right]\psi = 0$$

NOW THE "GAMMA" MATRICES:

$$\gamma^\mu = (\gamma^0,\ \gamma^1,\ \gamma^2, \gamma^3) \qquad \vec{\sigma} = (\sigma^1,\ \sigma^2,\ \sigma^3) = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

- Note that this is a particular representation of the matrices
- Any set of matrices satisfying the anti-commutation relations works

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- There are an infinite number of possibilities: this particular one (Björken-Drell) is just one example

IN FULL GLORY:

$$\left[i\hbar \begin{pmatrix} \frac{\partial}{\partial ct} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial ct} & -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & -\frac{\partial}{\partial ct} & 0 \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial ct} \end{pmatrix} - \begin{pmatrix} mc & 0 & 0 & 0 \\ 0 & mc & 0 & 0 \\ 0 & 0 & mc & 0 \\ 0 & 0 & 0 & mc \end{pmatrix} \right] \begin{pmatrix} \overbrace{\begin{matrix} \psi_1 \\ \psi_2 \end{matrix}}^{\psi_A} \\ \underbrace{\begin{matrix} \psi_3 \\ \psi_4 \end{matrix}}_{\psi_B} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p_\mu \Leftrightarrow i\hbar\partial_\mu$$

$$\begin{pmatrix} p_0 - mc & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -p_0 - mc \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

Consider applying another matrix to this equation

$$\begin{pmatrix} p_0 + mc & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -p_0 + mc \end{pmatrix} \begin{pmatrix} p_0 - mc & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -p_0 - mc \end{pmatrix} = \begin{pmatrix} p_0^2 - m^2c^2 - (\mathbf{p}\cdot\sigma)^2 & 0 \\ 0 & p_0^2 - m^2c^2 - (\mathbf{p}\cdot\sigma)^2 \end{pmatrix}$$

$$(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b})$$

$$(\sigma\cdot\mathbf{p})^2 = \mathbf{p}\cdot\mathbf{p}$$

$$\begin{pmatrix} p_0^2 - \mathbf{p}^2 - m^2c^2 & 0 \\ 0 & p_0^2 - \mathbf{p}^2 - m^2c^2 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- this is just the KG equation four times
    - Wavefunctions that satisfy the Dirac equation also satisfy KG

SOLUTIONS TO THE DIRAC EQUATION:

$$k^\mu = \frac{1}{\hbar}\,(E/c,\ p_x,\ p_y,\ p_z)$$

- Consider a particle at rest: $\psi(x) \sim e^{-ik\cdot x} = e^{\frac{-i}{\hbar}(\frac{E}{c}t - \mathbf{p}\cdot\mathbf{x})}$

  - Particle has no spatial dependence, only time dependence.

$$(i\hbar\gamma^0 \frac{\partial}{\partial ct} - mc)\psi = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t}\psi_A \\ \frac{\partial}{\partial t}\psi_B \end{pmatrix} = \frac{-imc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

  - Note that the equation breaks up into two independent parts:

$$\frac{\partial}{\partial t}\psi_A = -i\frac{mc^2}{\hbar}\psi_A \qquad -\frac{\partial}{\partial t}\psi_B = -i\frac{mc^2}{\hbar}\psi_B$$

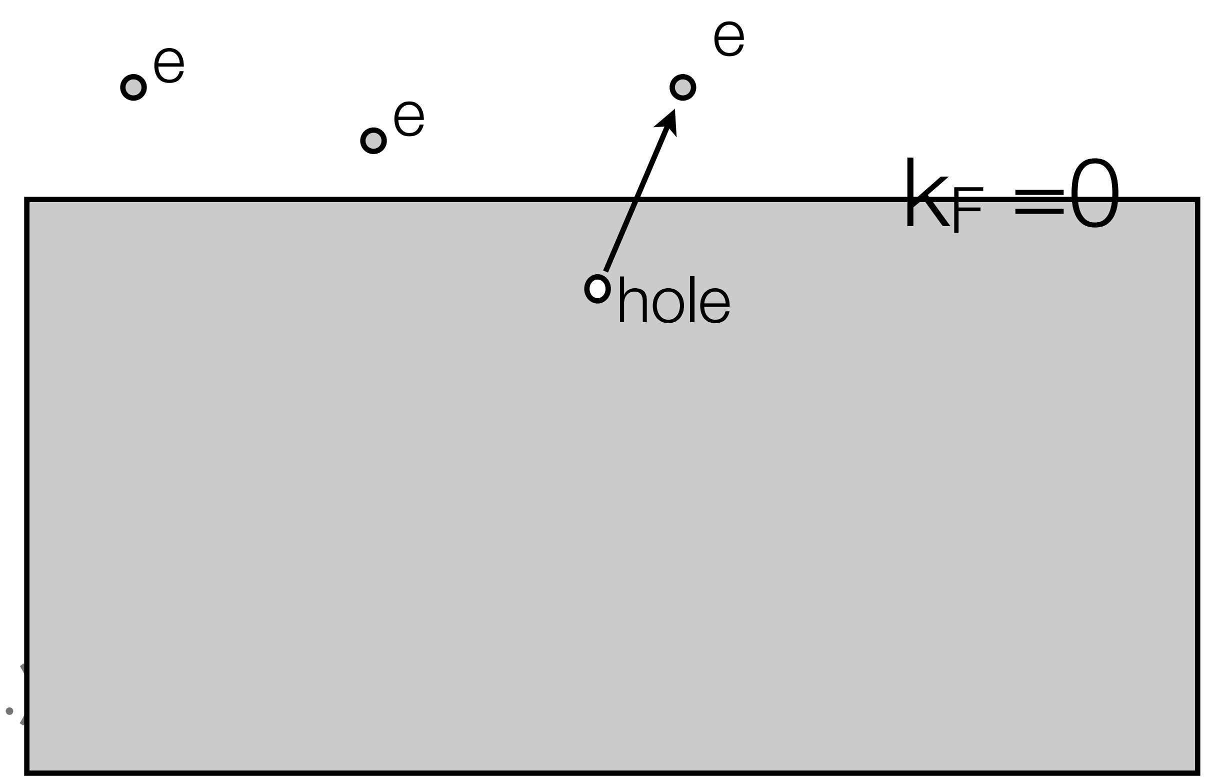
$$\psi_A(t) = e^{-i(\frac{mc^2}{\hbar})t}\psi_A(0) \qquad \psi_B(t) = e^{-i(-\frac{mc^2}{\hbar})t}\psi_B(0)$$

DIRAC'S DILEMMA:

- ψ_B appears to have negative energy

$$\psi_A(t) = e^{-i(\frac{mc^2}{\hbar})t}\psi_A(0) \qquad \psi_B(t) = e^{-i(-\frac{mc^2}{\hbar})t}\psi_B(0)$$

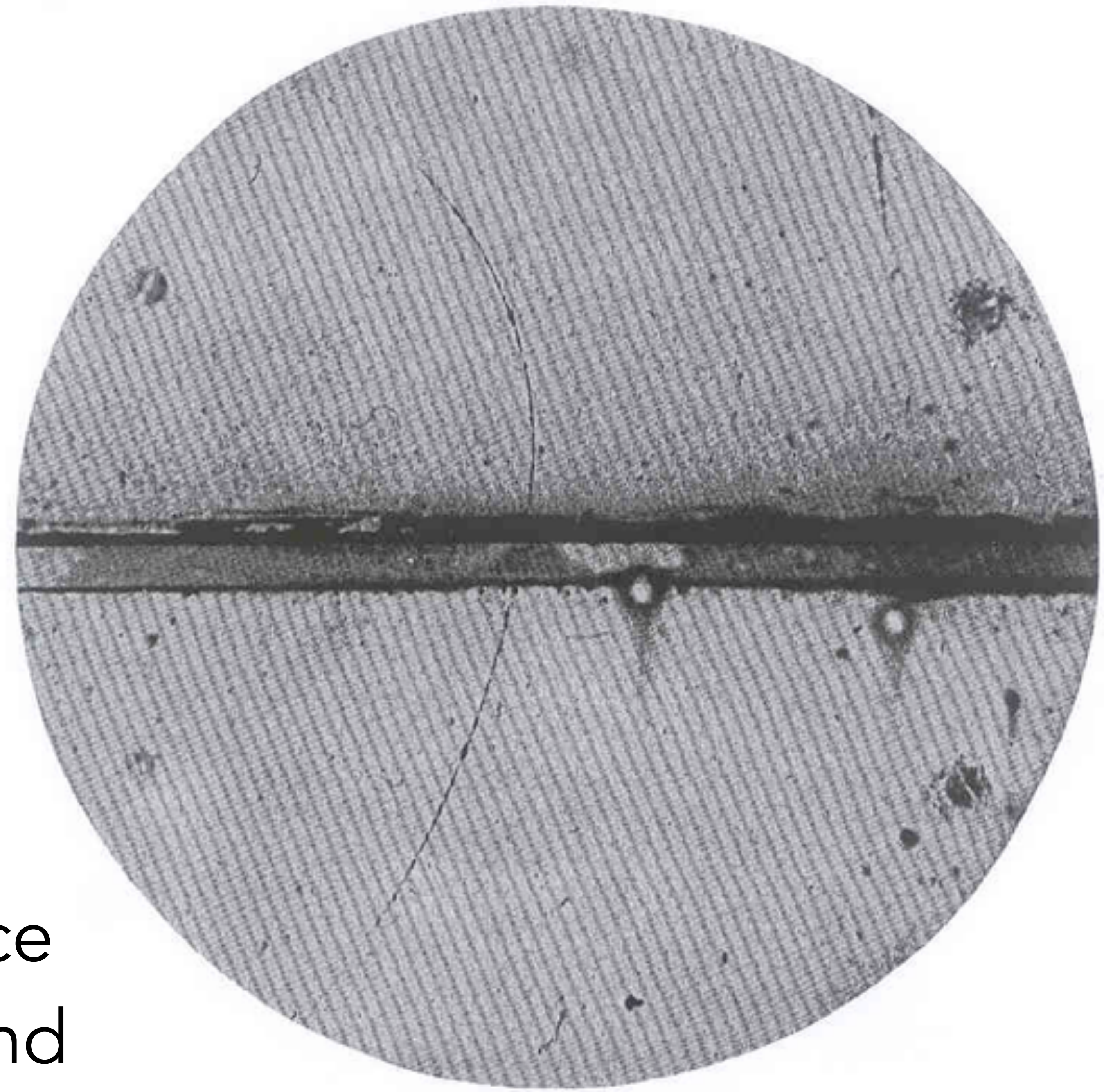
- Why don't all particles fall down into these states (and down to -∞)?
- Dirac's excuse: all electron states in the universe up to a certain level (say E=0) are filled.
- Pauli exclusion prevents collapse of states down to E = -∞
- We can "excite" particles out of the sea into free states
  This leaves a "hole" that looks like a particle with opposite properties (positive charge, opposite spin, etc.)



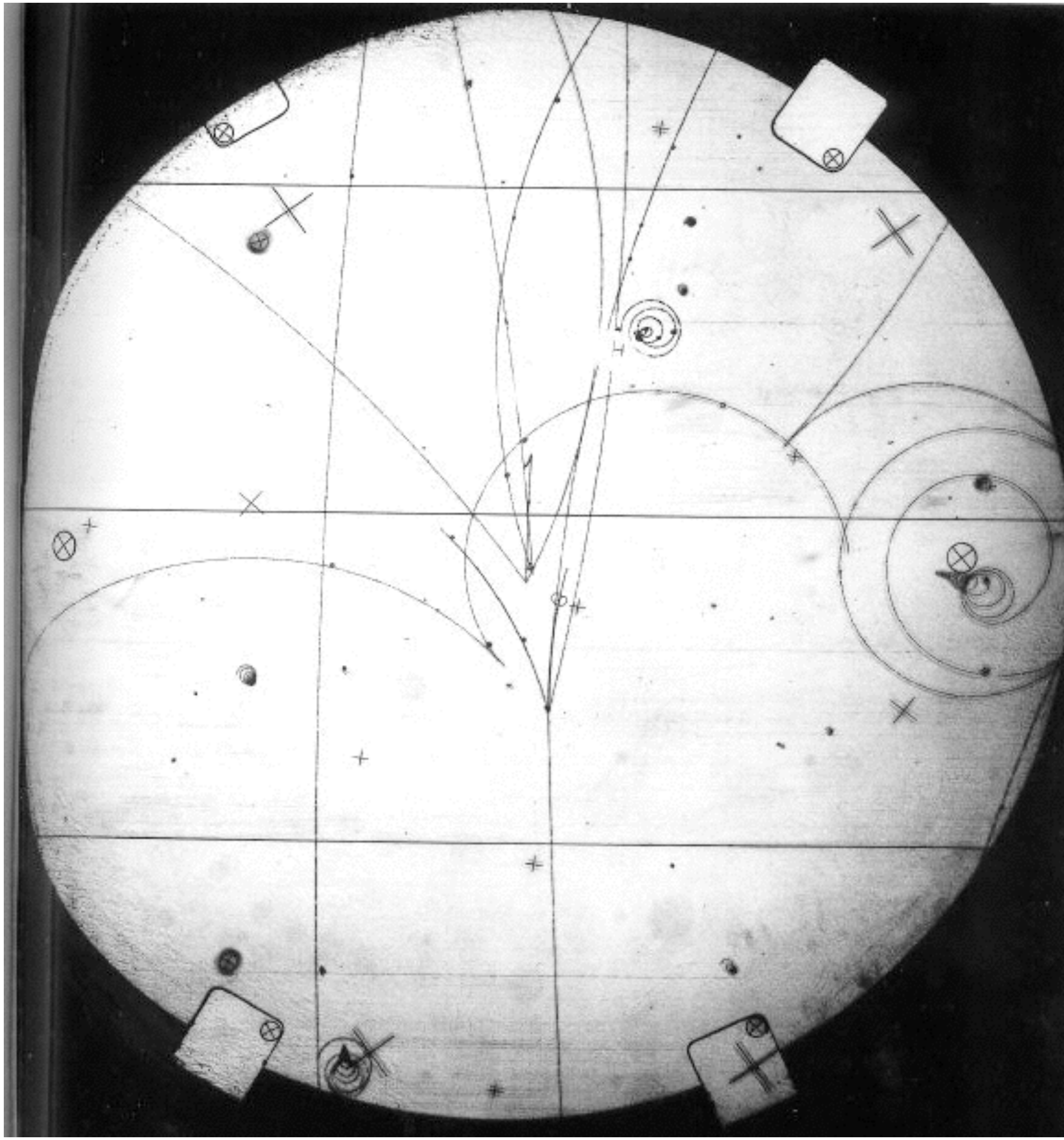
Dirac originally proposed that this might be the proton

EXCUSE TO TRIUMPH

- 1932: Anderson finds “positrons” in cosmic rays
- Exactly like electrons but positively charged:
  Fits what Dirac was looking for



Dirac predicts the existence of anti-matter and it is found



SOLUTIONS TO THE DIRAC EQUATION AT REST:

$$\psi_1(t) = e^{-imc^2t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi_2(t) = e^{-imc^2t/\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

"spin up" "spin down"

positive energy solutions (particle)

$$\psi_3(t) = e^{+imc^2t/\hbar} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi_4(t) = e^{+imc^2t/\hbar} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

"spin down" "spin up"

"negative" energy solutions (anti-particle)

- Note that all particles have the same mass

PEDAGOGICAL SORE POINT

- All the discussion we had thus far about difficulties with relativistic equations, (negative probabilities, negative energies) is of historic interest
- Scientifically, the framework for dealing with quantum mechanics and special relativity (i.e. quantum field theory) had not been developed
    - The old tools of NR quantum mechanics had reached their limit and new ones were necessary.
    - In particular, the idea of a “wavefunction” had to be revisited
    - Until this was done, there were many difficulties!
    - Once QFT was developed, all of these problems go away.
- Both KG and Dirac Equations are valid in QFT
    - No negative probabilities, no negative energies
- Nonetheless, the history and its course are rather interesting.

LORENTZ PROPERTIES:

- The Dirac equation "works" in all reference frames.
  - What exactly does this mean?

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0$$ "Lorentz Covariant"

- i, $\hbar$, m, γ, and c are constants that don't change with reference frames.
- ∂_μ and ψ will change with reference frames, however.
  - ∂_μ is a derivative that will be taken with respect to the space-time coordinates in the new reference frame. We'll call this ∂'_μ
  - how does ψ change?
    - $\psi' = S\psi$ where ψ' is the spinor in the new reference frame
- Putting this together, we have the following transformation of the equation when evaluating it in a new reference frame

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \quad \Rightarrow \quad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

What properties does S need to make this work?

TRANSFORMING THE DIRAC EQUATION:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \qquad \Rightarrow \qquad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

$$i\hbar\gamma^\mu\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu(S\psi) - mc\,(S\psi) = 0$$

S is constant in space time, so we can move it to the left of the derivatives

$$i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,(S\psi) = 0$$

Now slap S^{-1} from both sides

$$i\hbar\gamma^\nu\partial_\nu\psi - mc\,\psi = 0 \qquad S^{-1} \rightarrow i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,S\psi = 0$$

Since these equations must be the same, S must satisfy

$$\gamma^\nu = S^{-1}\gamma^\mu S\;\frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

EXAMPLE: THE PARITY OPERATOR

- For the parity operator, we want to invert the spatial coordinates while keeping the time coordinate unchanged:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\frac{\partial x_0}{\partial x_0{}'} = 1 \qquad \frac{\partial x_1}{\partial x_1{}'} = -1$$

$$\frac{\partial x_2}{\partial x_2{}'} = -1 \qquad \frac{\partial x_3}{\partial x_3{}'} = -1$$

- We then have

$$\gamma^0 = S^{-1}\gamma^0 S$$
$$\gamma^1 = -S^{-1}\gamma^1 S$$
$$\gamma^2 = -S^{-1}\gamma^2 S$$
$$\gamma^3 = -S^{-1}\gamma^3 S$$

Recalling

$$\gamma^\nu = S^{-1}\gamma^\mu S \, \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\gamma^0)^2 = 1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0$$

We find that γ^0 satisfies our needs

$$\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$
$$\gamma^i = -\gamma^0\gamma^i\gamma^0 = \gamma^0\gamma^0\gamma^i = \gamma^i$$

$$S_P = \gamma^0$$

LORENTZ COVARIANT QUANTITIES

- Recall that for four vectors a^μ, b^μ
  - a^μ, b^μ transform as Lorentz vectors (obviously)
  - $a^\mu b_\mu$ is a scalar (does not change under Lorentz transformations
  - $a^\mu b^\nu$ is a tensor (each has a Lorentz transformation)
- From the previous discussion, we know:
  - Dirac spinors have four components, but don't transform as Lorentz vectors
  - How do combinations of Dirac spinors change under Lorentz Transformations?

HOW DO WE CONSTRUCT A SCALAR?

- We can use γ^0: define: $\bar{\psi} = \psi^\dagger \gamma^0$
  - Consider a Lorentz transformation with S acting on the spinor
  - We can also show generally that $S^\dagger \gamma^0 S = \gamma^0$
  - This gives us $\bar{\psi}\psi \Rightarrow \psi^\dagger S^\dagger \gamma^0 S \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$
  - so this is a Lorentz invariant
- We can construct the parity operator to check how $\bar{\psi}\psi$ transforms under the parity operation.
  - Recall $S_P = \gamma^0$
    - We can investigate how $\bar{\psi}\psi$ transforms under parity

$$\bar{\psi}\psi \Rightarrow (\psi^\dagger S_P^\dagger \gamma^0)(S_P \psi) = \psi^\dagger \gamma^0 \gamma^0 \gamma^0 \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$$

$\bar{\psi}\psi$ doesn't change sign under parity
it is a Lorentz scalar

THE γ^5 OPERATOR

- Define the operator γ^5 as:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- It anticommutes with all the other γ matrices:

$$\{\gamma^\mu, \gamma^5\} = 0$$

  - use the canonical anti-commutation relations to move γ^μ to the other side
  - γ^μ will anti-commute with for $\mu \neq \nu$
  - γ^μ will commute when $\mu = \nu$
- We can then consider the quantity $\bar{\psi}\gamma^5\psi$
  - Can show that this is invariant under Loretnz transformation.
- What about under parity?

$$\bar{\psi}\gamma^5\psi \Rightarrow (\psi^\dagger S_P^\dagger)\gamma^0\gamma^5(S_P\psi) = -(\psi^\dagger S_P^\dagger)\gamma^0 S_P\gamma^5\psi = -\psi^\dagger\gamma^0\gamma^5\psi = -\bar{\psi}\gamma^5\psi$$

$\bar{\psi}\gamma^5\psi$ is a “pseudoscalar”

OTHER COMBINATIONS

- We can use γ^μ to make vectors and tensor quantities:

$\bar{\psi}\psi$	scalar	1 component	
$\bar{\psi}\gamma^5\psi$	pseudoscalar	1 component	
$\bar{\psi}\gamma^\mu\psi$	vector	4 components	
$\bar{\psi}\gamma^\mu\gamma^5\psi$	pseudovector	4 components	
$\bar{\psi}\sigma^{\mu\nu}\psi$	antisymmetric tensor	6 components	$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

- You can tell the transformation properties by looking at the Lorentz indices
  - γ^5 introduces a sign (adds a "pseudo")
  - Every combination of $\psi^*_i\psi_j$ is a linear combination of the above.
- We will see that the above are the basis for creating interactions with Dirac particles. Interactions will be classified as "vector" or "pseudovector", etc.

NEXT TIME:

- Please read 4.6-4.9
- I will not be in class on Thursday
  - Randy has kindly agreed to work out a phase space calculation
  - It would also be a good opportunity to ask questions, etc.