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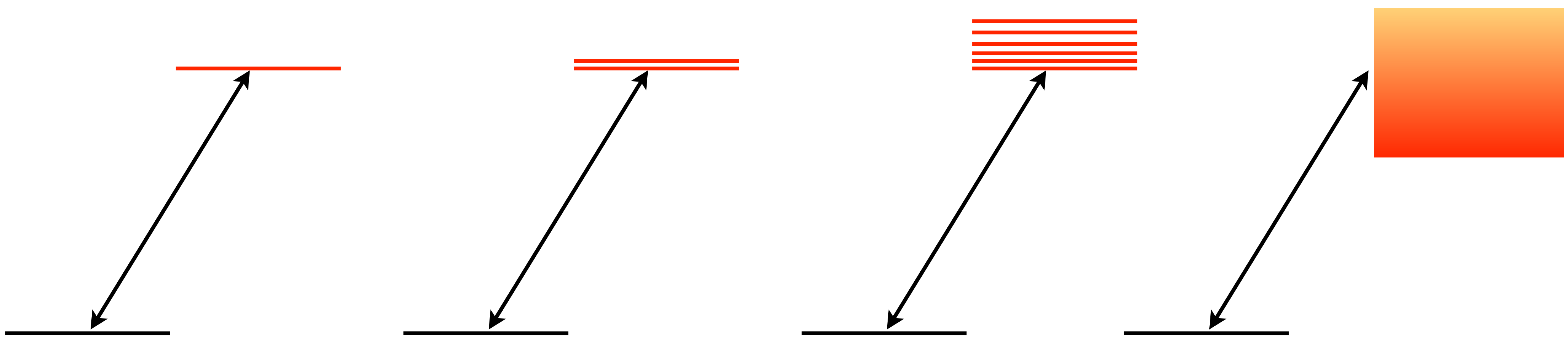
LECTURE 4:
PHASE SPACE IN DECAYS

GOLDEN RULE

- Fermi's golden rule states that the probability of a transition in quantum mechanics is given by the product of:
  - The absolute value of the matrix element (a k a amplitude) squared
  - The available density of states.

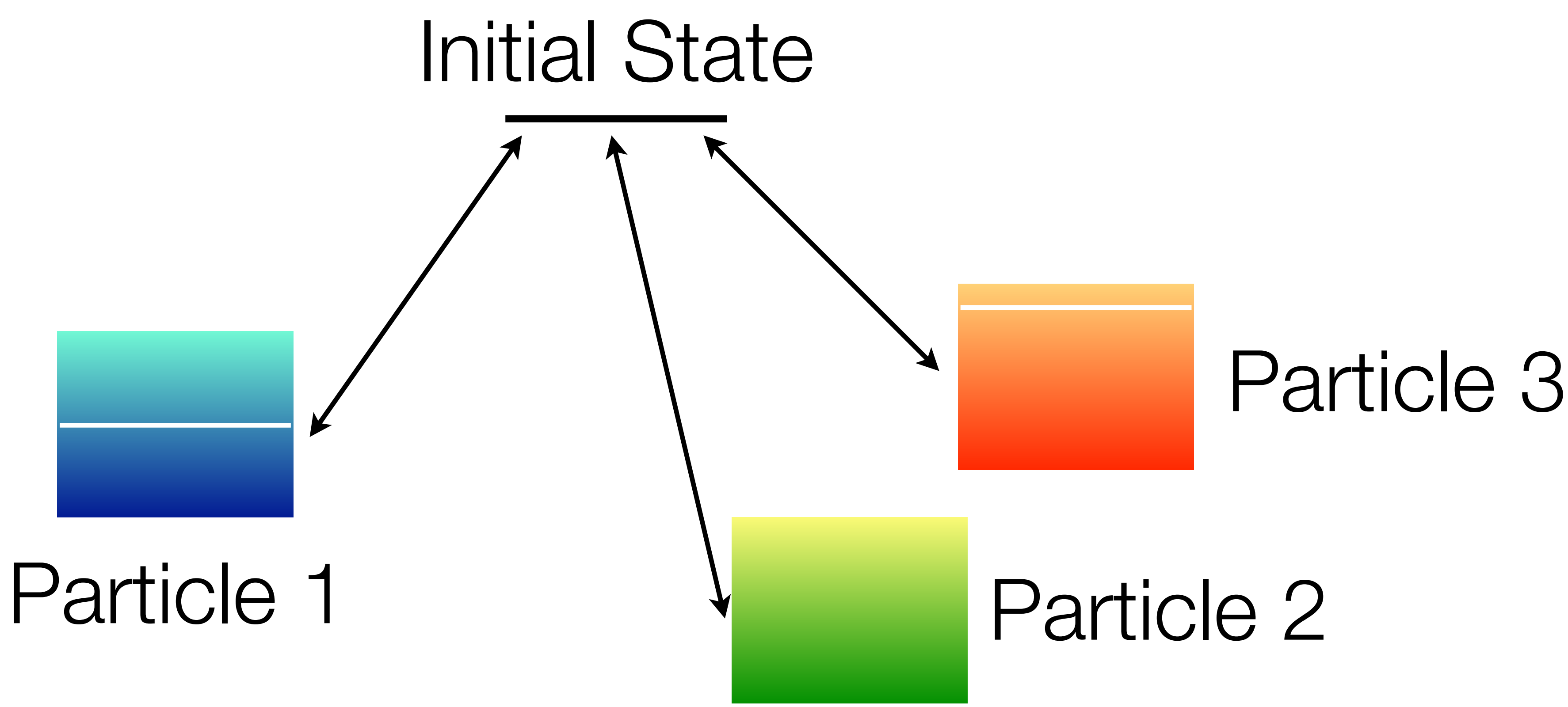
$$P \propto |\mathcal{M}|^2 \times \rho$$



$$P \propto |\mathcal{M}|^2 \qquad P \propto 2 \times |\mathcal{M}|^2 \qquad P \propto \int |\mathcal{M}(E)|^2 \, \rho(E) \, dE$$

- Typically a decay of a particle into states with lighter product masses has more "phase space" and more likely to occur.
- Let's see how to calculate the phase space
  - we'll learn how to calculate amplitudes later.

PRODUCT OF PHASE SPACE



- What is net phase space for the particle 1,2,3 to end up in particular places?
  - 0 if energy and momentum are not conserved
  - 0 if particles are not on "mass shell"
- Otherwise, the product of the individual phase spaces:

$$\rho = \rho_1(p_1^\mu) \times \rho_2(p_2^\mu) \times \rho_3(p_3^\mu)$$

integral extends over region satisfying kinematic constraints

$$\rho_{tot} = \int_{allowed} \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \rho_1(p_1^\mu)\rho_2(p_2^\mu)\rho_3(p_3^\mu)$$

PHASE SPACE IN DECAYS

$$\Gamma = \frac{S}{2\hbar m} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \; \delta^4\Big(\sum_i p_i^\mu - \sum_f p_f^\mu\Big) \times \prod_{j=2}^{N} 2\pi \; \delta(p_j^2 - m_j^2 c^2) \; \Theta(p_j^0) \; \frac{d^4 p_j}{(2\pi)^4}$$

- S: Symmetry factor
- $|\mathcal{M}|^2$: Matrix element factor (function of kinematics, polarizations, etc.)
- $\delta^4(\sum_i p_i^\mu - \sum_f p_f^\mu)$: Energy and momentum must be conserved
- $\prod_{j=2}^{N}$: Product over all outgoing particles
- $\delta(p_j^2 - m_j^2 c^2)$: Outgoing particles must be on mass shell
- $\Theta(p_j^0)$: Energy must be positive
- $\frac{d^4 p_j}{(2\pi)^4}$: distributed evenly in phase space

- Complicating looking, but represents a basic statement:
  - apart from matrix element, phase space is distributed evenly among all particles subject to mass requirements, E/p conservation
  - "dynamics" like parity violation, etc. incorporated into matrix element.

THE SYMMETRY FACTOR:

- Consider the integration of phase space for two particles in the final state where the particles are of the same species.

$$\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}$$

- At some point, say, there will be a configuration where $p_1 = K_1$ and $p_2 = K_2$
  - Since the particles are identical, we should also have the reverse case:
    - $p_1 = K_2$, $p_2 = K_1$
    - the integral will contain both cases separately.
  - However, in quantum mechanics, the identicalness of particles of the same species means that these are the same state and we have double counted.
  - We need to add a factor of 1/2 to the phase space
- Likewise, for n identical particles in the final state, we need a factor of 1/n!

THE GOLDEN RULE: 2-BODY DECAY

$$\Gamma = \frac{S}{2\hbar m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1 - p_2 - p_3)$$

$$\times(2\pi) \cdot \delta(p_2^2 - m_2^2 c^2)\Theta(p_2^0) \times (2\pi) \cdot \delta(p_3^2 - m_3^2 c^2)\Theta(p_3^0)$$

$$\frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4}$$

Let's integrate over overall outgoing particle phase space to get the total decay rate

$$d^4 p \equiv dp^0 dp^1 dp^2 dp^3$$

- Start with the phase space factors:

$$\delta(p^2 - m^2 c^2) = \delta((p^0)^2 - \vec{p}^2 - m^2 c^2)$$

$$\delta(p^2 - m^2 c^2) = \frac{1}{2 \times \sqrt{\vec{p}^{\,2} + m^2 c^2}} \left[\delta(p^0 - \sqrt{\vec{p}^{\,2} + m^2 c^2}) + \delta(p^0 + \sqrt{\vec{p}^{\,2} + m^2 c^2}) \right]$$

- Ignore the 2nd δ function since $\Theta(p^0)$ will be 0 whenever p^0 is negative

$$\delta(p^2 - m^2 c^2) = \frac{1}{2 \times \sqrt{\vec{p}^{\,2} + m^2 c^2}} \delta(p^0 - \sqrt{\vec{p}^{\,2} + m^2 c^2})$$

$$p_0 \Rightarrow \sqrt{\vec{p}^{\,2} + m^2 c^2}$$

ENFORCING ENERGY/MOMENTUM CONSERVATION

- Now integrate over p^0_3 and p^0_2 using the previous relations

$$\Gamma = \frac{S}{2\hbar m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \, \delta^4(p_1^\mu - p_2^\mu - p_3^\mu)$$

$$\times 2\pi \times \delta(p_2^2 - m_2^2 c^2)\Theta(p_2^0) \times 2\pi \times \delta(p_3^2 - m_3^2 c^2)\Theta(p_3^0)$$

$$\frac{d^3\vec{p}_2}{(2\pi)^3} \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{dp_2^0}{(2\pi)} \frac{dp_3^0}{(2\pi)}$$

$$\frac{1}{2 \times \sqrt{\vec{p_2}^{\,2} + m_2^2 c^2}} \qquad \frac{1}{2 \times \sqrt{\vec{p_3}^{\,2} + m_3^2 c^2}}$$

note p^0_2 and p^0_3 are now set according to E/p conservation by the δ function

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\vec{p_2}^{\,2} + m_2^2 c^2}\,\sqrt{\vec{p_3}^{\,2} + m_3^2 c^2}} \; d^3\vec{p}_2 \, d^3\vec{p}_3$$

DECAY AT REST:

- Decompose the product delta function (particle 1 at rest)

$$\delta^4(p_1^\mu - p_2^\mu - p_3^\mu) \Rightarrow \delta(m_1 c - \sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2} - \sqrt{\vec{p}_3^{\ 2} + m_3^2})\ \delta^3(-\vec{p}_2 - \vec{p}_3)$$

- Perform the d^3p_3 integral

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2}\ \sqrt{\vec{p}_3^{\ 2} + m_3^2 c^2}}\ d^3\vec{p}_2\ d^3\vec{p}_3$$

$$\downarrow$$

$$\sqrt{\vec{p}_2^{\ 2} + m_3^2 c^2}$$

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta(m_1 c - \sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2} - \sqrt{\vec{p}_2^{\ 2} + m_3^2 c^2})}{\sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2}\ \sqrt{\vec{p}_2^{\ 2} + m_3^2 c^2}}\ d^3\vec{p}_2$$

INTEGRAL IN SPHERICAL COORDINATES

$$d^3\vec{p}_2 \Rightarrow d\phi \; d\cos\theta \; |p_2|^2 dp_2$$

Assume no dependence of M on p

$$\Gamma = \frac{S}{32\pi^2\hbar m_1} \times \int d\phi \; d\cos\theta \; |\mathbf{p}_2|^2 \; d|\mathbf{p}_2| |\mathcal{M}|^2 \times \frac{\delta(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2 c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2 c^2})}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2} \; \sqrt{\mathbf{p}_2^2 + m_3^2 c^2}}$$

$$\int_0^{2\pi} d\phi \rightarrow 2\pi \qquad \int_{-1}^{+1} d\cos\theta \; \rightarrow 2$$

$$\begin{aligned} u &= \sqrt{\mathbf{p}_2^2 + m_2^2 c^2} + \sqrt{\mathbf{p}_2^2 + m_3^2 c^2} \\ du &= \frac{u|\mathbf{p}_2|}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}\sqrt{\mathbf{p}_2^2 + m_3^2 c^2}} d|\mathbf{p}_2| \end{aligned}$$

$$\Gamma = \frac{S}{8\pi\hbar m_1} \times \int_{p_2=0, u=m_2+m_3}^{\infty} du \; |\mathcal{M}|^2 \times \; \delta(m_1 c - u) \frac{|\mathbf{p}_2|}{u}$$

The final integral over u sends u=m_1c and makes p_2 consistent with E conservation

FINAL RESULT: TOTAL TWO-BODY DECAY RATE:

$$\Gamma = \frac{S}{8\pi\hbar m_1^2 c} \times |\mathcal{M}|^2 \times |\vec{p}_2|$$

- We now need to be able to calculate the matrix element M
  - We'll use Feynman diagrams and the associated calculus to calculate amplitudes for various elementary processes

SUMMARY

- Please read 4.1-4.5 for next time