PHYSICS 489/149

LECTURE 3: REVIEW OF QUANTUM MECHANICS

OFFICE HOURS:

- According to the doodle poll:
- Everyone can make it to either:
  - Monday at 4 pm
  - Tuesday at 4 pm
- Office hours will (usually) be held at this time

LAST TIME:

- We reviewed special relativity
  - we will mainly be interested in particle kinematics
    - energy, momentum, mass
  - importance of invariant quantities
  - pay attention to 3- vs. 4-vectors!
- Today, we move to quantum mechanics
  - review basic concepts in quantum dynamics
  - currents
  - spin and angular momentum
  - time dependent perturbation theory and scattering

BASIC QUANTUM MECHANICS

- The Schrödinger Equation:

$$\hat{H}\psi = i\dot{\psi} \qquad \hat{p} = -i\nabla$$

- for non-relativistic quantum mechanics

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x) \qquad \left[-\frac{1}{2m}\nabla^2 + \hat{V}\right]\psi = i\dot{\psi}$$

- Consider

$$|\psi|^2 = \psi^*\psi$$

$$\psi^* \rightarrow \quad -\frac{1}{2m}\nabla^2\psi = i\dot{\psi} \quad \leftarrow \psi^* \qquad \psi \rightarrow \quad -\frac{1}{2m}\nabla^2\psi^* = -i\dot{\psi}^* \quad \leftarrow \psi$$

$$-\frac{1}{2m}\left[\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right] = i(\psi^*\dot{\psi} + \psi\dot{\psi}^*)$$

$$-\frac{1}{2m}\nabla\cdot\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] = i\frac{\partial}{\partial t}\left[\psi^*\psi\right]$$

CONSERVED CURRENT

- conserved current:

$$\nabla \cdot \mathbf{j} + \dot{\rho} = \mathbf{0}$$

- Consider the previous equations:

$$-\frac{1}{2m}\nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*] = i\frac{\partial}{\partial t}[\psi^* \psi]$$

- we can consider this a conserved current with

$$\rho = |\psi|^2 \qquad \mathbf{j} = -i\frac{1}{2m}[\psi^* \nabla \psi - \psi \nabla \psi^*]$$

  - corresponding to the conserved flow of particle (density)

COMMUTATORS:

- $[A, B] = AB - BA$
- Convince yourself:
  - $[AB,C] = A[B,C] + [A,C]B$
  - $[A,BC] = [A,B]C + B[A,C]$
- Consequences for operators that commute?
- Canonical commutation relation
  - $[x,p] = i$
  - If we label $(x, y, z) \rightarrow (r_1, r_2, r_3)$, $(p_x, p_y, p_z) \rightarrow (p_1, p_2, p_3)$
    - $[r_a, p_b] = i\,\delta_{ab}$

ANGULAR MOMENTUM

- From classical mechanics:
  - **L** = **r** x **p**
    - $L_x = y\ p_z - z\ p_y$ . . . . . .
  - $\mathbf{L_i} = \boldsymbol{\varepsilon}_{ijk}\ r_j\ p_k$
    - $\boldsymbol{\varepsilon}_{ijk} = 0$ if any of ijk are equal
    - $\boldsymbol{\varepsilon}_{ijk} = +1$ if ijk is even permutation of 123
    - $\boldsymbol{\varepsilon}_{ijk} = -1$ if ijk is even permutation of 123
- From the canonical commutation relations:
  - $[L_i, L_j] = i\ \boldsymbol{\varepsilon}_{ijk}\ L_k$
  - $[L_x,L_y] = iL_z$ . . . . .
  - what consequences does this have for simultaneous eigenstates?
- Usually, we choose to diagonalize in L_z

TOTAL ANGULAR MOMENTUM

- We can consider the magnitude of the angular momentum
  - $L^2 = L_x^2 + L_Y^2 + L_Z^2$
  - $[L^2, L_x] = 0$
- "Ladder operator": $L_\pm = L_x \pm iL_y$
  - $[L_z, L_\pm] = \pm L_\pm$
  - $L^2 = L_- L_+ + L_z + L_z^2$
- Consider an eigenstates $|l,m\rangle$
  - l eigenvalue of L^2, m eigenvalue of L_z
  - $L_z L_\pm |l,m\rangle = (m\pm1) L_\pm |l,m\rangle$
  - $L^2 |l,m\rangle = l(l+1) |l,m\rangle$
- Representations of angular momentum
  - we can have states of total orbital angular momentum in integers
  - also half-integer states corresponding to spin (more on this later)
  - 2l+1 states corresponding for angular momentum l states.

THE PAULI MATRICES

- Define the matrices:

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Convince yourself that:
  - they satisfy the commutation relations $[S_i, S_j] = i\,\boldsymbol{\varepsilon}_{ijk} S_k$
  - the vectors $\binom{1}{0}, \binom{0}{1}$ are the eigenvectors of S_z with the appropriate eigenvalues
  - operators S_+ and S_- have the desired properties.
- all states of this system have the appropriate eigenvalue for a spin 1/2 system for the operator S^2.

TIME-DEPENDENT PERTURBATION

- "weakly" interacting system
  - most energy in free motion with small potential energy/interaction
  - $H = H_0 + V$
  - Assume we know eigenstates of H_0

$$H_0|\phi_j\rangle = E_j|\phi_j\rangle \qquad \langle\phi_j|\phi_k\rangle = \delta_{jk} \qquad |\psi(x,t)\rangle = \sum_k c_k(t)e^{-iE_kt}|\phi_k\rangle$$

  - Employing Schrödinger's equation:

$$H|\psi\rangle = i\frac{d}{dt}|\psi\rangle$$

$$\sum_j \left[E_j + V\right]e^{-iE_jt}c_j|\phi_j\rangle \qquad\qquad i\sum_k \left[\dot{c}_k - iE_kc_k\right]e^{-iE_kt}|\phi_k\rangle$$

$$\sum_j Ve^{-iE_jt}c_k|\phi_j\rangle = i\sum_k \dot{c}_ke^{-iE_kt}|\phi_k\rangle$$

FIRST ORDER:

- Now assume that we start in a specific state
  - $c_i(0) = 1$, $c_{j\neq i}(0) = 0$
  - $V \ll H_0$ so that $c_i(t) \sim 1 \gg c_{j\neq i}(t)$ for all t

$$\sum_j V e^{-iE_j t} c_k |\phi_j\rangle = i \sum_k \dot{c}_k e^{-iE_k t} |\phi_k\rangle$$

$$\langle\phi_f| \to V e^{-iE_i t}|\phi_i\rangle \qquad \sim \quad \langle\phi_f| \to i\sum_k \dot{c}_k e^{-iE_k t}|\phi_k\rangle$$

$$\dot{c}_f = -i\langle\phi_f|V|\phi_i\rangle e^{i(E_f - E_i)t}$$

- integrate in time to get the transition amplitude from i→f

$$c_f(T) = -i\int_0^T dt\ \langle\phi_f|V|\phi_i\rangle e^{i(E_f - E_i)t}$$

$$\Gamma_{fi} = \frac{P_{fi}}{T} = \frac{1}{T}c_f^*(T)c_f(T) = |\langle\phi_f|V|\phi_i\rangle|^2 \frac{1}{T}\int_0^T dt\ \int_0^T dt'\ e^{i(E_f - E_i)t} e^{-i(E_f - E_i)t'}$$

FERMI'S GOLDEN RULE

- We employ the "delta function":

$$\int dx\, e^{i(k-k')x} = 2\pi \times \delta(k-k')$$

$$\downarrow$$

$$\Gamma_{fi} = |\langle\phi_f|V|\phi_i\rangle|^2 \frac{1}{T}\int dt \int dt'\, e^{i(E_f-E_i)t} e^{-i(E_f-E_i)t'}$$

$$= 2\pi|\langle\phi_f|V|\phi_i\rangle|^2 \frac{1}{T}\int dt\, e^{i(E_f-E_i)t}\delta(E_f-E_i)$$

- δ function enforces energy conservation
    - integrate over energy, with $\rho(E_f)$ = number of states at E_f

$$= 2\pi \int dE_f\, \rho(E_f)|\langle\phi_f|V|\phi_i\rangle|^2 \frac{1}{T}\int dt\, e^{i(E_f-E_i)t}\delta(E_f-E_i)$$

$$\equiv 2\pi|\langle\phi_f|V|\phi_i\rangle|^2\rho(E_i)$$

DO IT AGAIN . . . ..

- We can use our new approximation to improve the original result

$$c_f(T) = -i\int_0^T dt\ \langle\phi_f|V|\phi_i\rangle e^{i(E_f - E_i)t}$$

$$\downarrow$$

$$\sum_j V e^{-iE_j t} c_k |\phi_j\rangle = i\sum_k \dot{c}_k e^{-iE_k t}|\phi_k\rangle$$

$$T_{fi} = \langle\phi_f|V|\phi_i\rangle + \sum_{j\neq i} \frac{\langle\phi_f|V|\phi_j\rangle\langle\phi_j|V|\phi_i\rangle}{E_i - E_k}$$

PARTICLE DECAYS

- A particle of a given type is identical to all others of its type
  - some probability to decay within an infinitesimal time period dt
  - Γ is independent of how "old" the particle is.
- For an ensemble of particles, the total rate of change is:

$$dN = -\Gamma\, N dt \quad \Rightarrow \quad N(t) = N_0 e^{-\Gamma t}$$

- The number of surviving particles follows:
  - wait for half of the particles to disappear: "half life"

$$\frac{N(t)}{N_0} = \frac{1}{2} = e^{-\Gamma t} \quad \Rightarrow \quad t_{1/2} = \frac{\log 2}{\Gamma} \qquad N_0 \equiv N(0)$$

  - wait for the number to decrease by a factor of e: "lifetime"

$$\frac{N(t)}{N_0} = \frac{1}{e} = e^{-\Gamma t} \quad \Rightarrow \quad \tau = \frac{1}{\Gamma}$$

COMBINING DECAY RATES:

- If there are several decay "modes" each with a given rate Γ_i, the total decay rate is given by the sum of all the rates:

$$\Gamma_{tot} = \sum_i \Gamma_i \quad \Rightarrow \quad \tau = \frac{1}{\Gamma_{tot}}$$

- If you are observing only one of these decay modes as a function of time, you will still see the number of particles diminish as the total decay rate

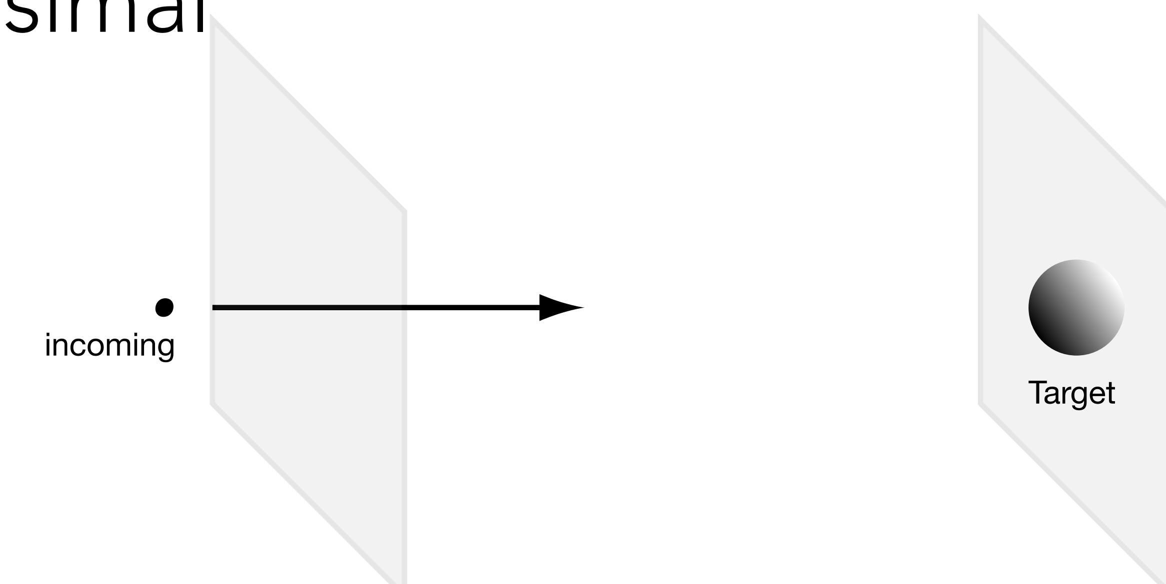
$$e^{-\Gamma_{tot} t} = e^{-t/\tau}$$

even though the rate of decay per unit time is a fraction of the total decay rate

  - You are observing a fraction of the total decays which means that the distribution will diminish as that fraction times the overall exponential.

SCATTERING RATES

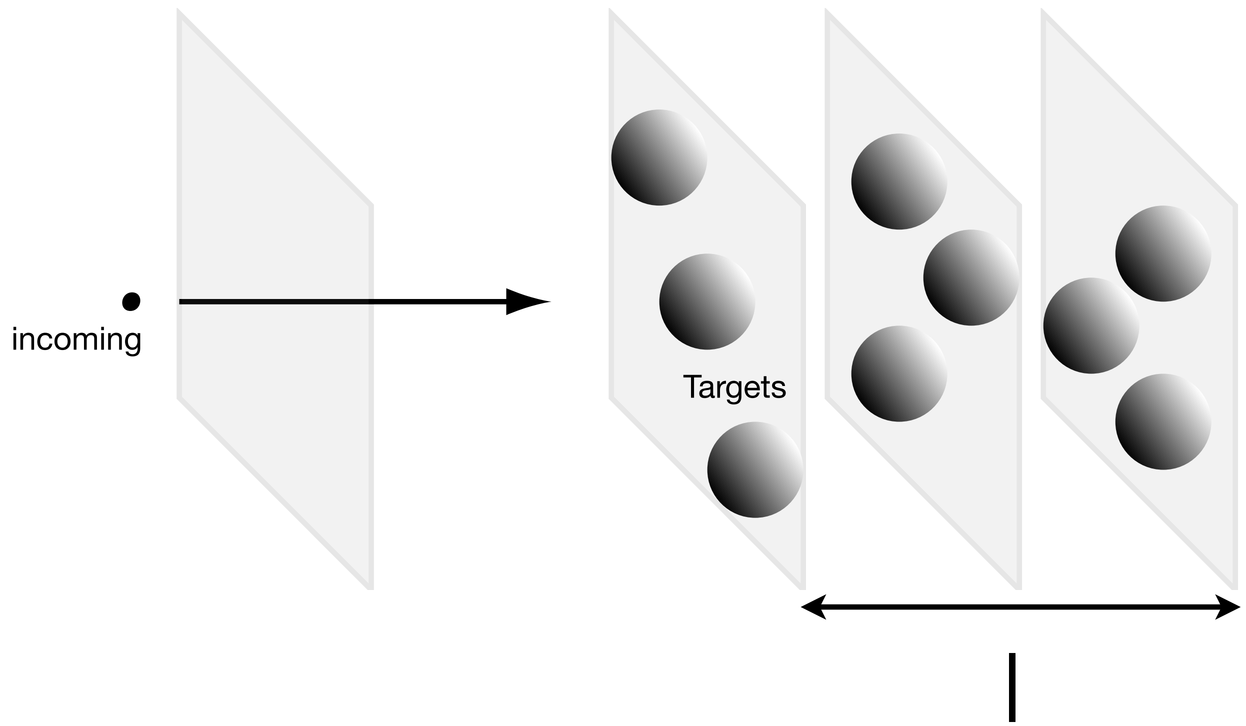
- Send in particles on a "target" and study what comes out
  - assume particles are "hard spheres" and projectile is infinitesimal



- Probability of interaction: area of target/unit area:
  - area of target particle = "cross section" σ
- Rate $\propto$ rate of incoming particles:
  - Luminosity $\mathcal{L}$ = particles/unit area/time

MORE THAN ONE TARGET

- More than one “layer” of target particles
- More than one target per unit area.



- Rate ∝ targets in the column swept by the incoming beam
  - Rate = N_T/Unit Area x σ x $\mathcal{L}$ = n l σ $\mathcal{L}$
  - n = number density of target particles, l = length of target

DIFFERENTIAL CROSS SECTION

- In hard sphere scattering, something "happening" is binary:
  - If the balls hit each other, then something happened
  - otherwise, nothing happened
- We generalize the idea of "something happening" by considering "differential cross section."
  - Probability that particle ends up in a particular part of phase space
  - e.g.. a particular momentum/angle range.

$$\sigma \Rightarrow \frac{d^3\sigma}{d\Omega\, dp}$$

polar angle

$$d\Omega = \sin\theta d\theta\ d\phi = d\cos\theta\ d\phi$$

"solid angle"

azimuthal angle

  - Notation lends itself to "integrating" over a phase space variable: say we don't care about the momentum but only the angle:

$$\frac{d\sigma}{d\Omega} = \int p^2 dp \frac{d^3\sigma}{d\Omega\, dp}$$

TOTAL CROSS SECTION

- "total cross section"
  - integrate over all phase space

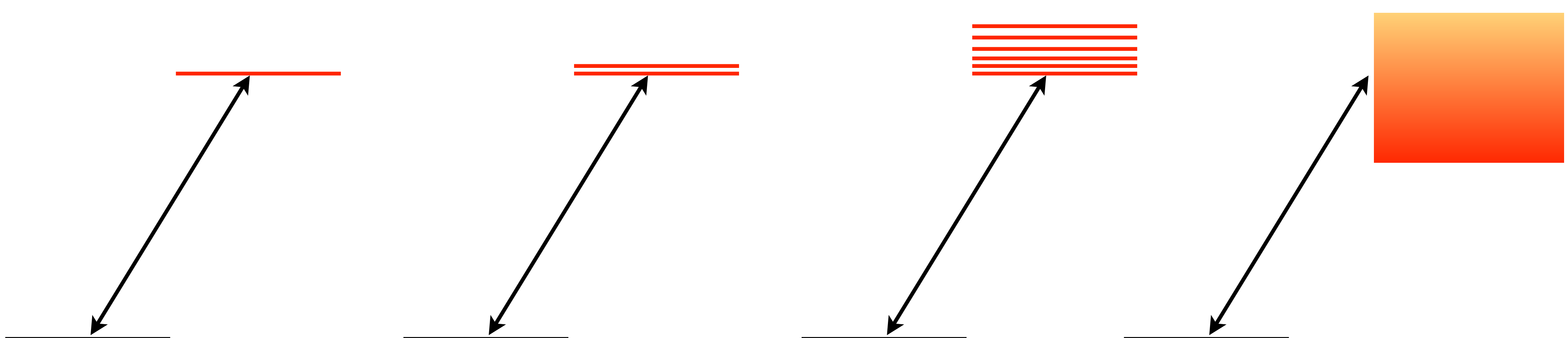
$$\sigma_{TOT} = \int p^2 dp \, d\phi \, d\cos\theta \frac{d^3\sigma}{d\Omega \, dp}$$

  - cross section for a particle to end up anywhere
- Note for "infinite range" interactions like the Coulomb interaction, the total cross section can be infinite; i.e. "something" always happens
  - This just reflects the fact that no matter how far you are away, there is still some electric field that will deflect your particle.

GOLDEN RULE:

- Fermi's golden rule states that the probability of a transition in quantum mechanics is given by the product of:
  - The absolute value of the matrix element (a k a amplitude) squared
  - The available density of states.

$$P \propto |\mathcal{M}|^2 \times \rho$$



$$P \propto |\mathcal{M}|^2$$

$$P \propto 2 \times |\mathcal{M}|^2$$

$$P \propto \int |\mathcal{M}(E)|^2 \, \rho(E) \, dE$$

- Typically a decay of a particle into states with lighter product masses has more "phase space" and more likely to occur.

SUMMARY:

- We reviewed basics of Quantum Mechanics
  - Schrödinger's Equation
  - Commutation relations
  - Angular Momentum
  - Fermi's Golden Rule rate of a process breaks down into
    - an amplitude
    - phase space/density of states factor
- Introduced basic concepts of rate in:
  - particle decays: decay rate and lifetimes
  - scattering: (differential cross sections)
- A few new mathematical objects:
  - Kronecker and Dirac δ
  - ε_{ijk}
  - Pauli matrices

NEXT TIME

- Please read Chapter 3

THE PAULI MATRICES

- Define the matrices corresponding to our S_i operators

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- eigenvectors corresponding to eigenstates of S, S_z.

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix} \qquad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

Dirac notation	Pauli matrix notation
$S_z\|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \frac{\hbar}{2}\|\tfrac{1}{2}, \tfrac{1}{2}\rangle$	$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$
$S_z\|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\frac{\hbar}{2}\|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$	$\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$

MORE ON PAULI MATRICES:

- Symbolic vs. Matrix form

$$S_+ = S_x + iS_y = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$S_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$S_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}\tfrac{3}{2} - \tfrac{1}{2}\tfrac{-1}{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_- = S_x - iS_y = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$S_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}\tfrac{3}{2} - \tfrac{-1}{2}\tfrac{1}{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

TOTAL ANGULAR MOMENTUM

$$S^2|\tfrac{1}{2},\tfrac{1}{2}\rangle = l(l+1)\hbar^2|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{3}{4}\hbar^2|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$S^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle = l(l+1)\hbar^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{3}{4}\hbar^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4} \times \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \right]$$

$$\frac{3\hbar^2}{4} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{3\hbar^2}{4} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3\hbar^2}{4} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{3\hbar^2}{4} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3\hbar^2}{4} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$