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PHYSICS 489/1489

LECTURE 2: SPECIAL RELATIVITY

REMINDERS

- Please fill out doodle poll for office hours
  - I would like to start next week
  - http://doodle.com/poll/ep2gxhxkrhmwttw827d6sq27/

- One clarification/reminder about interactions/vertices . . ..

OVERVIEW

- Review of central postulates of special relativity
- Review some consequences
- Introduce four-vector and index notation
- Develop Lorentz algebra
- Invariant quantities
- Energy/momentum conservation

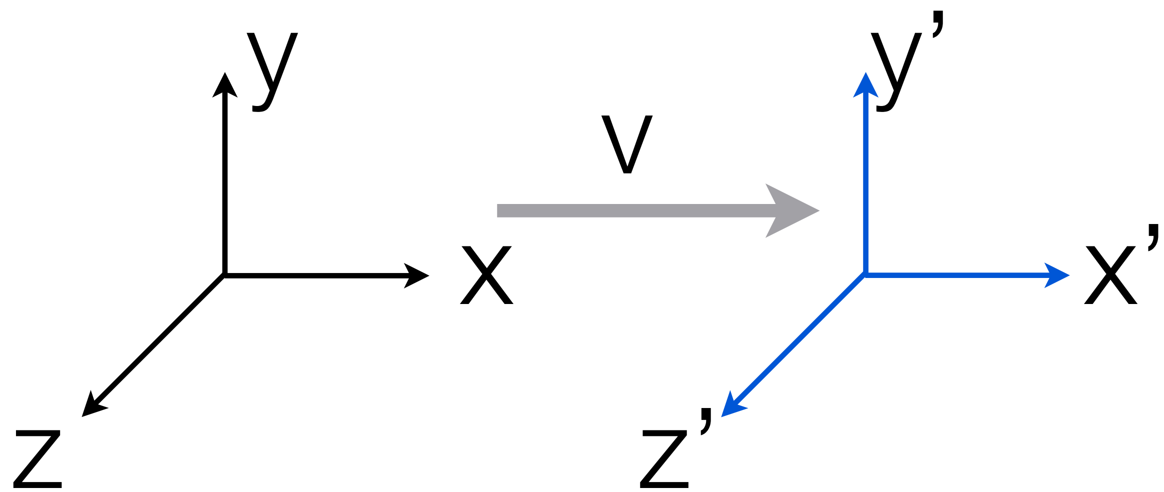
SPECIAL RELATIVITY

- Postulates:
  - the laws of physics are the same in all inertial reference frames
  - the velocity of light is the same in all reference frames
- Consequences:
  - simultaneity is relative; different according to reference frame
  - Lorentz (length) contraction
  - Time dilation
  - Strange velocity addition properties
    - speed of light is the same if you move towards it or away from it

LORENTZ TRANSFORMATION

- In 3D space, we know how coordinates "transform"
- There are corresponding transformations in SR
  - space and time coordinates w.r.t a frame moving with constant velocity to the original frame

$$\begin{aligned} t' &= \gamma(t - \tfrac{v}{c^2}x) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \qquad \begin{aligned} t &= \gamma(t' + \tfrac{v}{c^2}x') \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}$$



$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

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$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t'_A - t'_B = \gamma(t_A - t_B + \frac{v}{c^2}(x_B - x_A))$$

relativity of simultaneity and time dilation

$$x'_A - x'_B = \gamma(x_A - x_B + v(t_B - t_A))$$

length contraction

RELABELING . . . .

- Defining "4-vectors"
  - 3-vectors are objects like x,y, z components of something
  - they have definite transformation properties

$$
\begin{aligned}
t' &= \gamma(t - \tfrac{v}{c^2}x) & \qquad t &= \gamma(t' + \tfrac{v}{c^2}x') \\
x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\
y' &= y & y &= y' \\
z' &= z & z &= z'
\end{aligned}
$$

  - space-time 4-vector (ct, x, y, z) → (x^0, x^1, x^2, x^3)

$$
\begin{aligned}
x^{0'} &= \gamma(x^0 - \beta x^1) & \qquad x^0 &= \gamma(x'^0 + \beta x'^1) \\
x^{1'} &= \gamma(x^1 - \beta x^0) & x^1 &= \gamma(x'^1 + \beta x'^0) \\
x^{2'} &= x^2 & x^2 &= x'^2 \\
x^{3'} &= x^3 & x^3 &= x'^3
\end{aligned}
$$

MATRIX AND COMPONENT FORM

- We can write the transformations in matrix form . . .

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

- Or index form:
    - use indices to to express the matrix algebra:

$$x^{\mu\prime} = \sum_\nu \Lambda^\mu_\nu x^\nu$$

$$\Big\downarrow$$

$$x^{\mu\prime} = \Lambda^\mu_0 x^0 + \Lambda^\mu_1 x^1 + \Lambda^\mu_2 x^2 + \Lambda^\mu_3 x^3$$

    - note “upstairs” (superscript), “downstairs” (subscript) indices

SUMMATION NOTATION

- If two indices are repatedd with the same letter, summation "over" that index is implied

$$x^{\mu\prime} = \sum_{\nu} \Lambda^{\mu}_{\nu} x^{\nu} \rightarrow \Lambda^{\mu}_{\nu} x^{\nu}$$

- related index = "contracted", non-repeated "free"
- Define:
  - "contravariant" 4-vector: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$
  - "covariant" 4-vector: $x_0 = ct$, $x_1 = -x$, $x_2 = -y$, $x_3 = -z$
- Summation is always over covariant and contravariant indices
  - just a way to keep track of that "minus" sign
- The index notation is insensitive to the ordering of terms

THE METRIC TENSOR

- contra/covariant 4-vectors are related by:

$$x^\mu = g^{\mu\nu} x_\nu \qquad\qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- another way to keep track of the "minus sign"
- "Metric":

$$g^{\mu\nu} x_\mu x_\nu \rightarrow x_\mu x^\mu$$

  - invariant under Lorentz transformations
    - same in all reference frames
    - what is the analog for a three vector?

GENERALIZATION

- We can take two four vectors and take their product

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} \ldots$$

  - this is also invariant with respect to Lorentz transformations
- Indices tell us how to classify quantities by how they transform
  - invariant/"scalar": no free indices, do not "transform" or depend on reference frame
  - "vector": one free index:

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$$

  - "tensor": one L for each free index:

$$\Lambda_1{}^{\mu\nu\prime} \rightarrow \Lambda_2{}^\mu_\rho \Lambda_2{}^\nu_\sigma \Lambda_1{}^{\rho\sigma} \qquad x^{\mu\prime} y^{\nu\prime} = \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho y^\sigma$$

ENERGY-MOMENTUM:

- We can construct the "energy/momentum" 4-vector:

$$\tau = \frac{t}{\gamma}$$

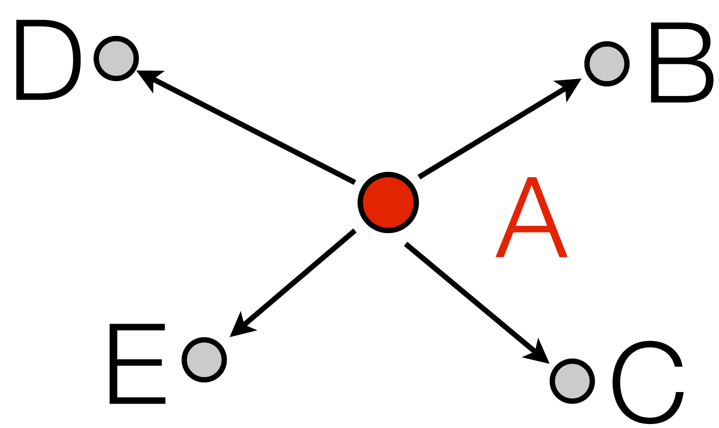
  - take derivatives of the space-time vector wrt. τ:

$$\eta^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma(\frac{d(ct)}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = \gamma(c, \vec{v})$$

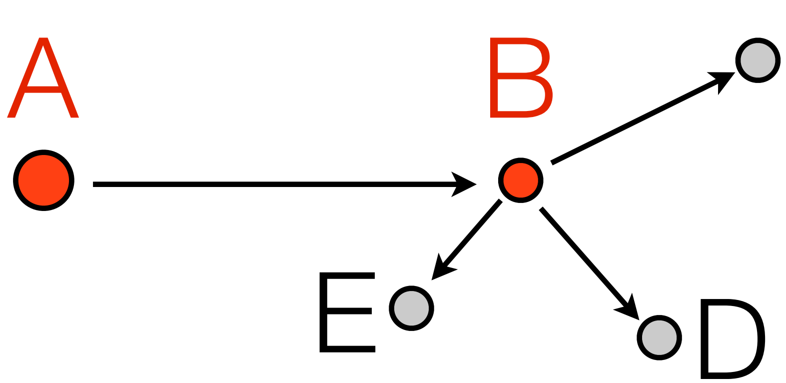
  - can show that $\eta^{\mu}\eta_{\mu} = c^2 \rightarrow \eta^{\mu}$ is a 4-vector
- Multiply η^{μ} by the mass of the particle to define p^{μ}
  - $p^{\mu} = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p})$
  - defines the energy/momentum of an object with invariant product mc^2
  - each component is a conserved quantity
- Examples of other 4-vectors?

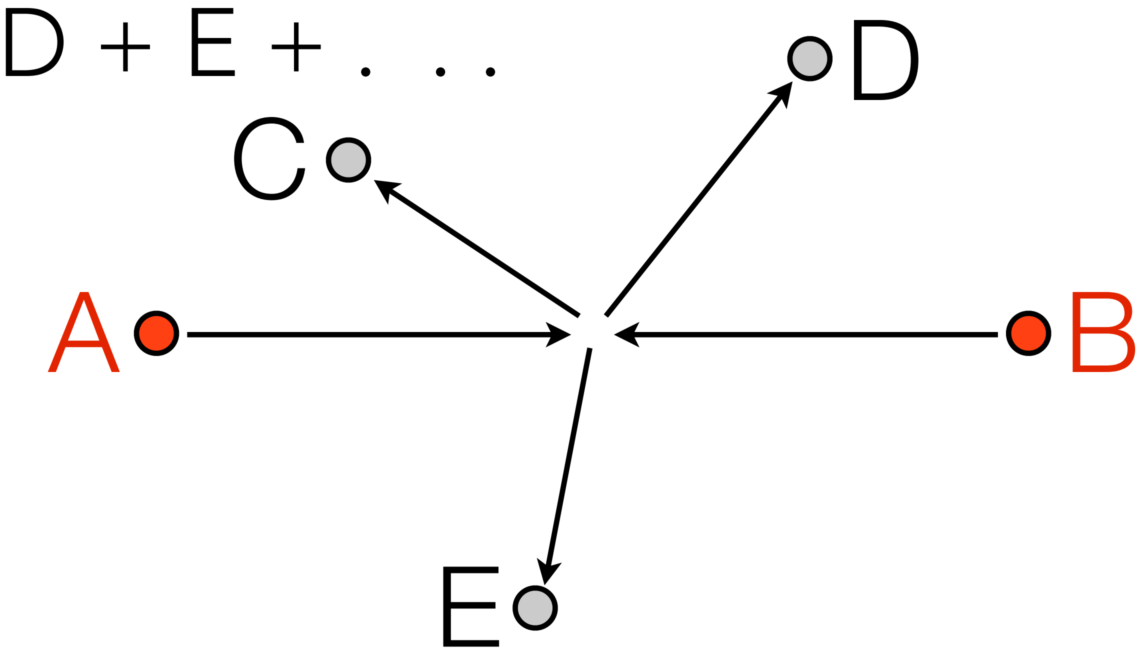
DECAYS AND SCATTERS

- Decays: A → B+C



- Scattering: A + B→ C + D + E + . . .





CONSERVATION

Four equations relating the initial and final state energies and momenta

- Energy conservation

$$\sum_i E_I^i = \sum_j E_F^j$$

$$\sum_i {p^i}_I^{\mu} = \sum_j {p^j}_F^{\mu}$$

- Momentum Conservation

$$\sum_i {p^i_y}_I = \sum_j {p^j_y}_F$$

$$\sum_i \vec{p}_I^{\,i} = \sum_j \vec{p}_F^{\,j}$$

INVARIANTS

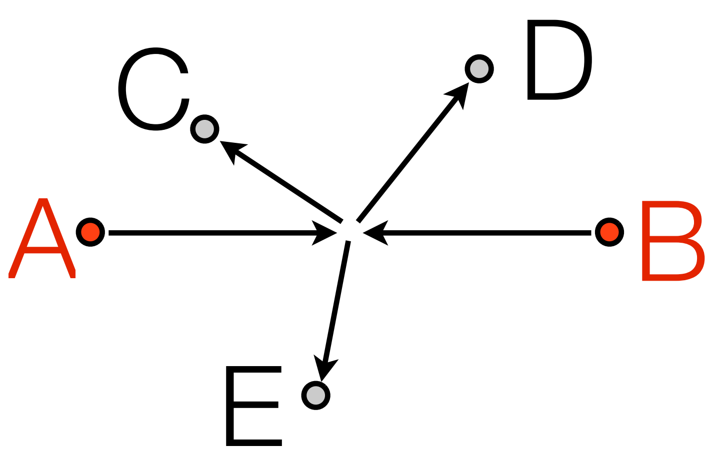
- "dot" product of two 4-vectors to make a scalar:
  - $a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$
  - $= a^\mu b_\mu$, $a_\mu b^\mu$, $g_{\mu\nu} a^\mu b^\nu$, etc.
- Explicitly in terms of two -4momentum vectors:
  - $p_1 \cdot p_2 = p_1{}^0 p_2{}^0 - p_1{}^1 p_2{}^1 - p_1{}^2 p_2{}^2 - p_1{}^3 p_2{}^3 = E_1 E_2 / c^2 - \mathbf{p_1} \cdot \mathbf{p_2}$
    - the dot product of a four momentum with itself:
    - $p_1 \cdot p_1 = p_1{}^2 = (E_1/c)^2 - \mathbf{p_1}^2 = \ . \ . \ . \ .$
- Invariants:
  - are the same in all reference frames
  - reduces multicomponent equation to scalar quantities
  - they may save you from having to explicitly evaluate by choosing a reference frame/coordinate frame
  - If massless particles are involved, it will eliminate terms

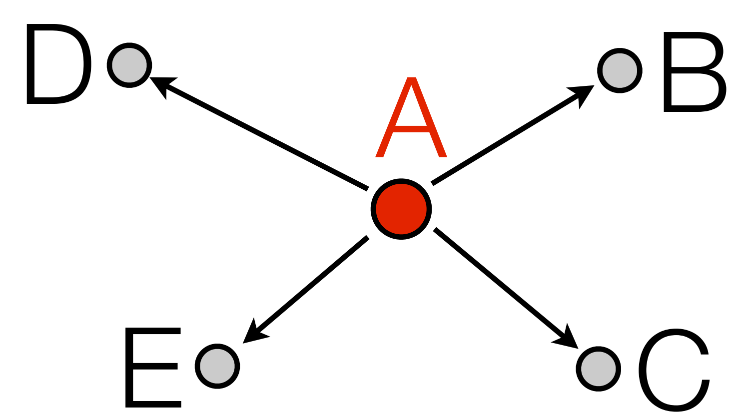
REFERENCE FRAMES:

- We will typically operate in two kinds of reference frames:
- Centre-Of-Momentum

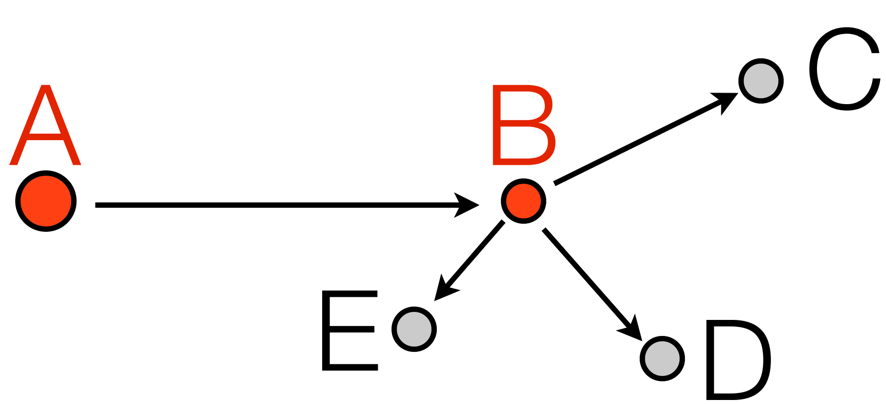
$$\sum_i \vec{p}_I^{\,i} = 0$$

  - sum of momentum is zero
  - decay of particle at rest, colliding beams

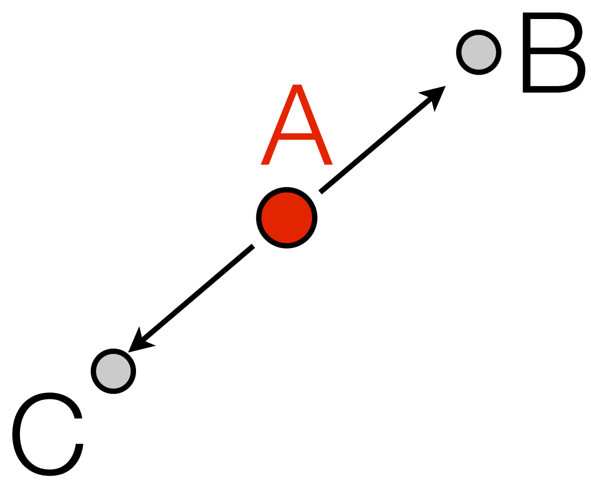




  - Lab frame: scattering with one particle at rest



SETTING UP THE KINEMATICS

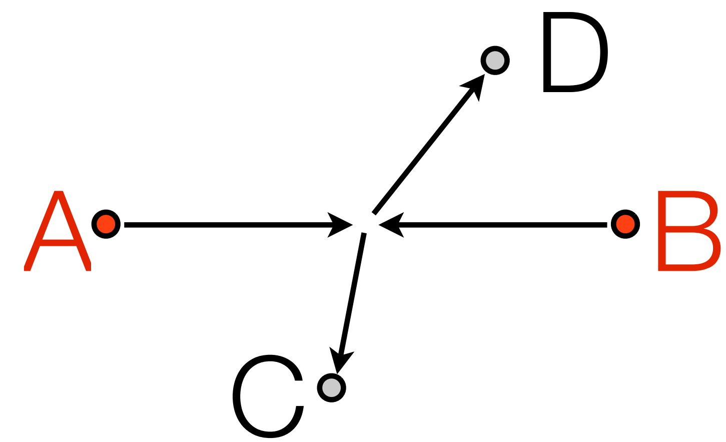


$$p_A = p_B + p_C$$

$$p_A^2 = (p_B + p_C)^2$$

$$= p_B^2 + p_C^2 + 2p_A \cdot p_B$$

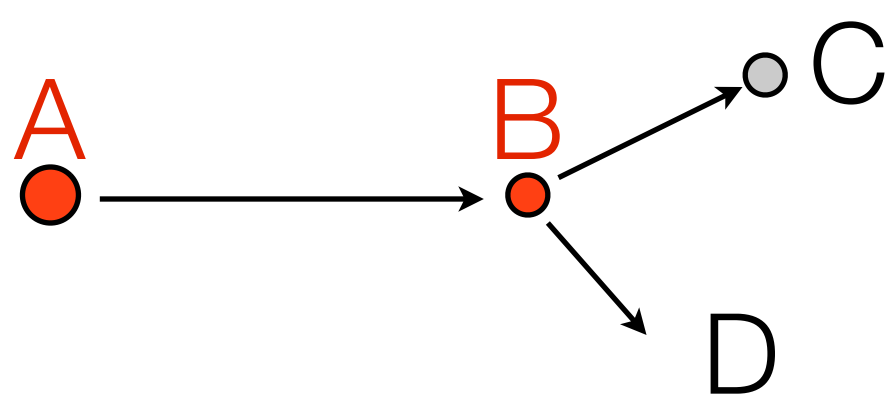
$$m_A^2 c^2 = m_B^2 c^2 + m_C^2 c^2 + 2p_A \cdot p_B$$



$$p_A + p_B = p_C + p_D$$

$$(p_A + p_B)^2 = (p_C + p_D)^2$$

$$m_A^2 c^2 + m_B^2 c^2 + 2p_A \cdot p_B = m_C^2 + m_D + 2p_C \cdot p_D$$



$$p_A + p_B = p_C + p_D$$

$$p_A + p_B - p_C = p_D$$

$$(p_A + p_B - p_C)^2 = p_D^2$$

$$m_A^2 c^2 + m_B^2 c^2 + m_C^2 c^2 + 2p_A \cdot p_B - 2p_A \cdot p_C - 2p_B \cdot p_C = m_D^2 c^2$$

LABORATORY-FRAME SCATTERING

- Consider the process A + B → C where B is at rest.
  - What energy of A required to produce C?
  - Assign labels (trivial: A → p_A, b → p_B, C → p_C)
  - Conservation of 4-momentum:

$$p_A + p_B = p_C$$

  - Square the equation:

$$(p_A + p_B)^2 = p_C^2 \qquad p_A^2 + p_B^2 + 2(p_A \cdot p_B) = p_C^2$$

$$m_A^2 c^2 + m_B^2 c^2 + 2(p_A \cdot p_B) = m_C^2 c^2$$

  - Now in the lab frame:

$$p_A = (E_A/c, \mathbf{p}_A)$$

$$p_B = (m_B c, \mathbf{0})$$

$$p_A \cdot p_B = E_A m_B$$

$$\boxed{E_A = \frac{m_C^2 c^2 - m_A^2 c^2 - m_B^2 c^2}{2 m_B}}$$

APPLICATION:

- What minimum energy is required for the reaction:

$$p + p \rightarrow p + p + p + \bar{p}$$

  - with one of the initial protons at rest to proceed?
  - in the lab frame, it is complicated since the outgoing products must be in motion to conserve momentum
  - in the CM frame, however, the minimum energy configuration is where the outgoing products are at rest.
    - (kinematically equivalent A+B→C, C=a single particle of mass $4m_p$)

$$E_1 = \frac{m_3^2c^2 - m_1^2c^2 - m_2^2c^2}{2m_2}$$

    - set $m_3 = 4m_p$, m_1,$m_2 = m_p$
    - $E_1 = 7\, m_pc^2$
    - Note “conservation” vs. “invariance”

COMPTON SCATTERING:

- Consider the process $\gamma + e \rightarrow \gamma + e$ where the electron is initially at rest.
  - If the γ scatters by an angle θ, what is it's outgoing energy?
  - Assign labels:
    - p_1 = incoming photon, p_2 = initial electron
    - p_3 = outgoing photon, p_4 = outgoing electron
  - Conservation of 4-momentum:

$$p_1 + p_2 = p_3 + p_4 \qquad\qquad p_1 + p_2 - p_3 = p_4$$

  - Square the equation:

$$(p_1 + p_2 - p_3)^2 = p_4^2 \qquad\qquad p_1^2 + p_2^2 + p_3^2 + 2(p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) = p_4^2$$

$$m_e^2 c^2 + 2(p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) = m_e^2 c^2$$

  - Now in the lab frame:

$$\begin{aligned} p_1 &= (E_1/c, \mathbf{p}_1) & p_1 \cdot p_2 &= E_1 m_e \\ p_2 &= (m_e c, \mathbf{0}) & p_1 \cdot p_3 &= E_1 E_3/c^2 - \mathbf{p}_1 \cdot \mathbf{p}_3 = E_1 E_3 (1 - \cos\theta)/c^2 \\ p_3 &= (E_3/c, \mathbf{p}_3) & p_2 \cdot p_3 &= E_3 m_e \end{aligned}$$

TIPS/HINTS

- Start by setting up 4-momentum conservation equation
- Typically, it will be helpful to square the expression at some point
  - reduce from 4 equations to one
  - note that squared 4-momentum is the mass of the particle!
  - move expressions around based on what quantity you are after and specifics of reference frame (which particle(s) are at rest, etc0.
- Be sure to keep track of what is
  - 4-momentum
  - 3-momentum
  - energy
- Make use of invariants if possible
  - quantities which are independent of reference frame

SUMMARY

- Review of special relativity postulates
- Introduce 4-vectors:
  - time + spatial coordinates
  - energy + momentum
- Index notation to represent “tensor” algebra
  - summation convention
  - formation of invariant quantities
- Kinematics using 4-momentum
  - 4-momentum conservation
  - use of invariant quantities
  - Frames: centre-of-momentum vs. laboratory

FOR NEXT TIME:

- Please read Chapter 2.3
- Please respond to doodle poll by the end of this week
  - I would like to start office hours next week