PHYSICS 489/1489

LECTURE 12: SYMMETRIES

ANNOUNCEMENTS

- Problem set 2 due today at 1700
  - please note again notes provided by Randy which are on the course website
- Midterm next Thursday (3 November)
  - covers material up to chapter 7
  - Short questions on Feynman diagrams, phase space, basic properties of electromagnetic, weak, strong interactions
  - One question with amplitude/cross section calculation

SYMMETRY

- An operation on something that leaves it unchanged
- Mathematically, symmetries form "groups"
  - closure: one operation followed by another is another symmetry operation
  - identity: doing nothing is a symmetry operation
  - inverse: for each operation, there is another symmetry operation that undoes it.
  - associativity: $O_1(O_2O_3) = (O_1O_2)O_3$
- Noether's theorem:
  - symmetry in a system ↔ conservation law

NOETHER'S THEOREM IN QM

- We can express the operation on a state as an operator:

$$|\psi\rangle \to U|\psi\rangle \qquad |U\psi\rangle \equiv U|\psi\rangle$$

- in order for the physical predictions to be unchanged by the operation, it must preserve:
    - normalization $\langle\psi|\psi\rangle \to \langle U\psi|U\psi\rangle$

$$\langle U\psi|U\psi\rangle \to \langle\psi|U^\dagger U\psi\rangle$$

        - can see that U must be unitary, i.e. $U^\dagger U = 1$
- eigenvalues of operators
    - particularly the Hamiltonian $[H, U] = 0$

CONTINUOUS GROUPS

- A continuous group is one that can be parameterized by continuous parameter(s):

$$U \to U(\theta)$$

- Examples:
  - rotations ("special" orthogonal matrices)
    - e.g. matrices where $O^T O = O O^T = 1$
    - with determinant 1
    - "SO(N)": special orthogonal matrices of dimension N
  - "special unitary" matrices:
    - unitary matrices with determinant 1
    - "SU(N)": special unitary matrices of dimension N

GENERATORS

- For a continuous group, we can consider an infinitesimal transformation (in a Taylor expansion sense)

$$U(\epsilon) = 1 + i\epsilon G + \mathcal{O}(\epsilon^2) + \dots .$$

- The operator G is called a “generator” of the group
- The unitarity of U requires G to be Hermitian
    - $G = G^\dagger$
- Since the infinitesimal transformation is an element of the group
    - $[H, G] = 0$
- Noether’s theorem in quantum mechanics:
    - The observable corresponding to G is conserved

GLOBAL GAUGE SYMMETRY:

- From Electromagnetism, we have "gauge" transformations:
  - Maxwell's laws are invariant under:

$$\phi = A_0 \rightarrow A_0 - \dot{\chi} \qquad \mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \qquad A_\mu \rightarrow A_\mu - \partial_\mu\chi$$

- Consider the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$
  - if we rotate the phase of ψ through all of space-time

$$\psi \rightarrow e^{i\theta}\psi$$

  - the Dirac equation remains valid (just an overall phase)
  - (n.b. May be easier to see if we consider the Lagrangian

$$\mathcal{L} = i\hbar\,\bar{\psi}\gamma^\mu\partial_\mu\psi - mc\,\bar{\psi}\psi \qquad \begin{array}{l}\psi \rightarrow e^{i\theta}\psi \\ \bar{\psi} \rightarrow e^{-i\theta}\bar{\psi}\end{array}$$

LOCAL GAUGE TRANSFORMATION

- Now consider a more radical transformation:
  - Adjust the phase of the field as a function of space time
  - i.e. θ becomes a function of x

$$\theta \to \theta(x) \qquad e^{-i\theta} \Rightarrow e^{-i\theta(x)}$$

  - this is called a "local gauge transformation"
- now consider the Dirac equation
  - $\partial_\mu \psi \Rightarrow \partial_\mu (e^{i\theta(x)}\psi) = e^{i\theta}(\partial_\mu \psi) + i(\partial_\mu \theta)e^{i\theta}\psi$
  - $(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow e^{i\theta} \times \left[i\partial\!\!\!/ - (\partial\!\!\!/\theta) - m\right]\psi = 0$
- extra term in the equation!

LOCAL GAUGE SYMMETRY

- Promote local gauge transformations to a symmetry
  - we *require* the equation to be invariant under local gauge transformations (i.e. space-time dependent phase rotations)
- The symmetry/invariance can be restored if:
  - we add a term to the equation

$$(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (i\partial\!\!\!/ - qA\!\!\!/ - m)\psi = 0$$

  - where:

$$A_\mu \rightarrow A_\mu - \frac{1}{q}\partial_\mu \theta(x) \qquad \psi \rightarrow e^{i\theta(x)}\psi$$

$$(i\partial\!\!\!/ - qA\!\!\!/ - m)\psi = 0$$

$$\Rightarrow e^{i\theta(x)} \times \left[i\partial\!\!\!/ - (\partial\!\!\!/\theta) - qA\!\!\!/ + \partial\!\!\!/\theta - m\right]\psi = 0$$

WHAT HAPPENED:

- We required the Dirac equation to be invariant under local gauge transformation
- this introduced a new field A with its own transformation
- Note:
  - A is a "vector" particle: i.e. A_μ
  - its transformation is the same as the EM gauge transformation
  - it couples to the Dirac field with a strength controlled by q
  - (it must be massless to preserve the symmetry)
- It has all the properties of a photon interacting with a Dirac particle with charge q
- electromagnetism is a "U(1) local gauge theory"

LINGO:

- "gauge symmetry" = "gauge invariance":
  - generalization of "phase symmetry"
- "covariant derivative":

$$\partial_\mu \to \partial_\mu + iqA_\mu$$

- "gauge boson"
  - vector field introduced for local gauge invariance
- "gauge theory"
  - particle system that has a gauge symmetry

GENERALIZATION:

- Consider the group SU(2)
  - "2x2 unitary matrices with determinant 1"
  - we can parameterize the group as follows:

$$U = U(\vec{\theta}) = e^{i\frac{g}{2}\vec{\theta}\cdot\vec{\sigma}}$$

  - where **σ** are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Note:
  - we can consider **σ** as generators of the group
  - there are three parameters which parametrize the group
  - the matrices act on two-component vectors/spinors

"NON-ABELIAN GAUGE SYMMETRY"

- Now consider a theory in which we postulate:
  - "local gauge invariance under SU(2)"
  - θ parameters become space-time dependent

$$\vec{\theta} \to \vec{\theta}(x)$$

  - the "space/objects" we act on have two component

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to e^{i\frac{g}{2}\vec{\theta}\cdot\vec{\sigma}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

  - for Dirac particles, this means that we are considering a pair of Dirac fields (each with 4 components) that transform under SU(2) operators

GAUGE INVARIANCE

- Like before, gauge invariance requires new fields because of the space-time dependence of θ(x)

- so we introduce three fields (A^i) that transform as follows:

$$\vec{A}_\mu \to \vec{A}_\mu - \frac{1}{2}\partial_\mu \theta - \frac{g}{2}\epsilon_{ijk}\theta_i A^j_\mu$$

- one can show that this preserves the invariance under the SU(2) transformations for this equation of motion

$$(i\not\partial - \frac{g}{2}\vec{\sigma} \cdot \vec{\not A} - m)\psi = 0$$

- the extra term in the A transformation results from the fact that the the σ matrices do not commute.

WHAT HAPPENED:

- We now have a system of two fermions, each described by the Dirac equation
- local gauge symmetry requires three fields to cancel the “leftover” terms from the transformation
    - we have three new gauge fields which mediate interactions
$$\frac{g}{2}\vec{\sigma} \cdot \vec{\not{A}}\psi$$
    - the additional term
$$\frac{g}{2}\vec{\theta} \times \vec{A}_\mu$$
        - leads to interactions between the gauge bosons themselves
            - i.e. the bosons are “charged”

FOUNDATIONS:

- Quantum field theory arises from
  - special relativity
  - quantum mechanics
- with Local Gauge Invariance
  - we introduce interactions via bosons required to maintain the symmetry
  - however, the bosons must be massless
- Note the difference between:
  - the symmetry group (i.e. possible operations)
  - the objects on which they act
  - imposing symmetries on their behaviour (equations of motion, Lagrangian, etc.)

NEXT TIME

- Please read 10.5, 10.6
  - You can skip 10.5.1 unless it helps you understand 10.5.2