ANNOUNCEMENTS

• Problem set 2 due today at 1700
  • please note again notes provided by Randy which are on the course website
• Midterm next Thursday (3 November)
  • covers material up to chapter 7
  • Short questions on Feynman diagrams, phase space, basic properties of electromagnetic, weak, strong interactions
  • One question with amplitude/cross section calculation
SYMMETRY

• An operation on something that leaves it unchanged
• Mathematically, symmetries form “groups”
  • closure: one operation followed by another is another symmetry operation
  • identity: doing nothing is a symmetry operation
  • inverse: for each operation, there is another symmetry operation that undoes it.
  • associativity: $O_1(O_2O_3) = (O_1O_2)O_3$
• Noether’s theorem:
  • symmetry in a system $\leftrightarrow$ conservation law
NOETHER’S THEOREM IN QM

- We can express the operation on a state as an operator:

  \[ |\psi\rangle \rightarrow U|\psi\rangle \quad |U\psi\rangle \equiv U|\psi\rangle \]

- In order for the physical predictions to be unchanged by the operation, it must preserve:
  - Normalization \( \langle \psi|\psi \rangle \rightarrow \langle U\psi|U\psi \rangle \)
    \[
    \langle U\psi|U\psi \rangle \rightarrow \langle \psi|U^\dagger U\psi \rangle
    \]
  - Can see that U must be unitary, i.e. \( U^\dagger U = 1 \)

- Eigenvalues of operators
  - Particularly the Hamiltonian \( [H, U] = 0 \)
CONTINUOUS GROUPS

• A continuous group is one that can be parameterized by continuous parameter(s):

\[ U \rightarrow U(\theta) \]

• Examples:
  
  • rotations ("special" orthogonal matrices)
    • e.g. matrices where \( O^T O = O O^T = 1 \)
    • with determinant 1
  
  • “SO(N)”: special orthogonal matrices of dimension N

  • “special unitary” matrices:
    • unitary matrices with determinant 1
    • “SU(N)”: special unitary matrices of dimension N
GENERATORS

- For a continuous group, we can consider an infinitesimal transformation (in a Taylor expansion sense)
  \[ U(\epsilon) = 1 + i\epsilon G + \mathcal{O}(\epsilon^2) + \ldots \]
- The operator G is called a "generator" of the group
- The unitarity of U requires G to be Hermitian
  - \[ G = G^\dagger \]
- Since the infinitesimal transformation is an element of the group
  - \[ [H, G] = 0 \]
- Noether’s theorem in quantum mechanics:
  - The observable corresponding to G is conserved
GLOBAL GAUGE SYMMETRY:

• From Electromagnetism, we have “gauge” transformations:

  • Maxwell’s laws are invariant under:

\[
\phi = A_0 \rightarrow A_0 - \dot{\chi} \quad A \rightarrow A + \nabla \chi \quad A_\mu \rightarrow A_\mu - \partial_\mu \chi
\]

• Consider the Dirac equation \((i\not{\partial} - m)\psi = 0\)

  • if we rotate the phase of \(\psi\) through all of space-time

\[\psi \rightarrow e^{i\theta} \psi\]

  • the Dirac equation remains valid (just an overall phase)

  • (n.b. May be easier to see if we consider the Lagrangian

\[
\mathcal{L} = i\hbar \bar{\psi} \gamma^\mu \partial_\mu \psi - mc \bar{\psi} \psi
\]

\[\psi \rightarrow e^{i\theta} \psi \quad \bar{\psi} \rightarrow e^{-i\theta} \bar{\psi}\]
LOCAL GAUGE TRANSFORMATION

• Now consider a more radical transformation:
  • Adjust the phase of the field as a function of space time
  • i.e. \( \theta \) becomes a function of \( x \)
  \[
  \theta \rightarrow \theta(x) \quad e^{-i\theta} \Rightarrow e^{-i\theta(x)}
  \]
  • this is called a "local gauge transformation"

• now consider the Dirac equation
  • \( \partial_\mu \psi \Rightarrow \partial_\mu (e^{i\theta(x)}\psi) = e^{i\theta} (\partial_\mu \psi) + i(\partial_\mu \theta)e^{i\theta} \psi \)
  • \((i\partial - m)\psi = 0 \Rightarrow e^{i\theta} \times [i\partial - (\partial \theta) - m] \psi = 0\)
  • extra term in the equation!
LOCAL GAUGE SYMMETRY

- Promote local gauge transformations to a symmetry
  - we require the equation to be invariant under local
gauge transformations (i.e. space-time dependent phase
corrections)

- The symmetry/invariance can be restored if:
  - we add a term to the equation
    \[(i\phi - m)\psi = 0 \Rightarrow (i\phi - q\hat{A} - m)\psi = 0\]
  - where:
    \[A_\mu \rightarrow A_\mu - \frac{1}{q}\partial_\mu \theta(x) \quad \psi \rightarrow e^{i\theta(x)}\psi\]
    \[(i\phi - q\hat{A} - m)\psi = 0\]
    \[\Rightarrow e^{i\theta(x)} \times [i\phi - (\partial\theta) - q\hat{A} + \phi\theta - m] \psi = 0\]
WHAT HAPPENED:

• We required the Dirac equation to be invariant under local gauge transformation
• this introduced a new field A with its own transformation
• Note:
  • A is a “vector” particle: i.e. $A_\mu$
  • its transformation is the same as the EM gauge transformation
  • it couples to the Dirac field with a strength controlled by $q$
  • (it must be massless to preserve the symmetry)
• It has all the properties of a photon interacting with a Dirac particle with charge $q$
• electromagnetism is a “U(1) local gauge theory”
LINGO:

• "gauge symmetry" = "gauge invariance":
  • generalization of "phase symmetry"

• "covariant derivative":
  \[ \partial_{\mu} \rightarrow \partial_{\mu} + iqA_{\mu} \]

• "gauge boson"
  • vector field introduced for local gauge invariance

• "gauge theory"
  • particle system that has a gauge symmetry
GENERALIZATION:

- Consider the group SU(2)
  - “2x2 unitary matrices with determinant 1”
  - we can parameterize the group as follows:
    \[ U = U(\vec{\theta}) = e^{i \frac{g}{2} \vec{\theta} \cdot \vec{\sigma}} \]
  - where \( \vec{\sigma} \) are Pauli matrices

Note:

- we can consider \( \vec{\sigma} \) as generators of the group
- there are three parameters which parametrize the group
- the matrices act on two-component vectors/spinors

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]
Now consider a theory in which we postulate:

- "local gauge invariance under SU(2)"
- $\theta$ parameters become space-time dependent

\[ \vec{\theta} \rightarrow \vec{\theta}(x) \]

- the "space/objects" we act on have two component

\[
\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow e^{i \frac{g}{2} \vec{\theta} \cdot \vec{\sigma}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

- for Dirac particles, this means that we are considering a pair of Dirac fields (each with 4 components) that transform under SU(2) operators
GAUGE INVARIANCE

• Like before, gauge invariance requires new fields because of the space-time dependence of $\theta(x)$

• so we introduce three fields $(A^i)$ that transform as follows:

$$\vec{A}_\mu \to \vec{A}_\mu - \frac{1}{2} \partial_\mu \theta - \frac{g}{2} \epsilon_{ijk} \theta_i A^j_\mu$$

• one can show that this preserves the invariance under the SU(2) transformations for this equation of motion

$$(i\not{\partial} - \frac{g}{2} \vec{\sigma} \cdot \vec{A} - m)\psi = 0$$

• the extra term in the $A$ transformation results from the fact that the $\sigma$ matrices do not commute.
WHAT HAPPENED:

• We now have a system of two fermions, each described by the Dirac equation

• local gauge symmetry requires three fields to cancel the “leftover” terms from the transformation

• we have three new gauge fields which mediate interactions

\[ \frac{g}{2} \vec{\sigma} \cdot \vec{A} \psi \]

• the additional term

\[ \frac{g}{2} \vec{\theta} \times \vec{A}_\mu \]

• leads to interactions between the gauge bosons themselves

• i.e. the bosons are “charged”
FOUNDATIONS:

- Quantum field theory arises from
  - special relativity
  - quantum mechanics
- with Local Gauge Invariance
  - we introduce interactions via bosons required to maintain the symmetry
  - however, the bosons must be massless
- Note the difference between:
  - the symmetry group (i.e. possible operations)
  - the objects on which they act
  - imposing symmetries on their behaviour (equations of motion, Lagrangian, etc.)
NEXT TIME

• Please read 10.5, 10.6

• You can skip 10.5.1 unless it helps you understand 10.5.2