PHY489/1489

LECTURE 10:
SPIN SUMMATION
ELECTRON-QUARK SCATTERING

SPIN SUMMATION

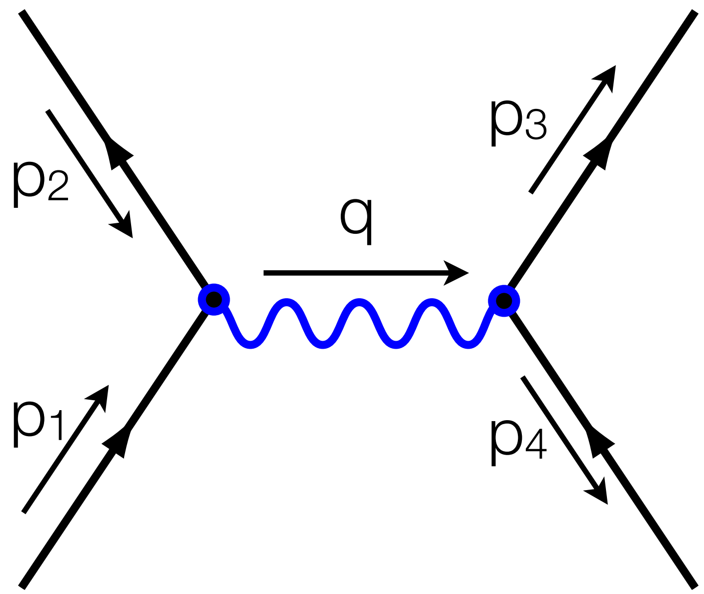
- Last time, we calculated amplitudes using the explicit form of the spinors
  - attractive features: straightforward if tedious math
  - this will always work; it may just be very tedious . ..
- Introduce: spin-summation
  - make use of general spinor properties to derive rather simple expressions
  - preserves Lorentz invariance, easier to deal with masses
  - disadvantage: some new math
- You should have some idea of how both work
  - some things will be (much) easier in one or the other approach.

"COMPLETENESS" RELATIONS

$$\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\mu p_\mu + mc) \qquad \sum_{s=1,2} v^s \bar{v}^s = (\gamma^\mu p_\mu - mc)$$

- "outer" product of two Dirac spinors
- sum over spin states s
  - there are two terms in each sum
- Note that these sums are exactly the same as when we:
  - summed over the outgoing spins in a reaction
    - if our detector doesn't measure the outgoing spin/helicity
  - averaged over incoming spins if the beam is unpolarized
    - this involves summing over all incoming spin configurations and dividing by the total number of configurations
    - e.g. divide by 4 for a pair of incoming unpolarized fermions

AMPLITUDE STRUCTURE



$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\ \gamma^\mu\ v(4)]\,[\bar{v}(2)\ \gamma_\mu\ u(1)]$$

- need to take the absolute value squared of $\mathscr{M}$

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{g_e^4}{(p_1+p_2)^4}\,[\bar{u}_3\gamma^\mu v_4]\,[\bar{v}_2\gamma_\mu u_1] \quad \times \quad [\bar{u}_3\gamma^\nu v_4]^*\,[\bar{v}_2\gamma_\nu u_1]^*$$

- Generally, we'll encounter expressions like this:

$$[\bar{u}_a\Gamma_1 u_b][\bar{u}_a\Gamma_2 u_b]^*$$

$$[\bar{u}_a\Gamma_1 u_b][\bar{u}_a\Gamma_2 u_b]^*$$

$$[\bar{u}_a\Gamma_1 v_b][\bar{u}_a\Gamma_2 v_b]^*$$

$$[\bar{v}_a\Gamma_1 u_b][\bar{v}_b\Gamma_2 u_b]^*$$

$$\begin{aligned}
[\bar{u}_a\Gamma_2 u_b]^* &= [\bar{u}_a\Gamma_2 u_b]^\dagger \\
&= [u_a^\dagger\gamma^0\Gamma_2 u_b]^\dagger = [u_b^\dagger\Gamma_2^\dagger\gamma^{0\dagger}u_a] \\
&= [u_b^\dagger\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u_a] \\
&= [\bar{u}_b\bar{\Gamma}_2 u_a] \ = [\bar{u}_b\Gamma_2 u_a]
\end{aligned}$$

COMPLETENESS RELATION

- We need to calculate terms like this

$$[\bar{u}_a \Gamma_1 u_b][\bar{u}_a \Gamma_2 u_b]^* = [\bar{u}_a \Gamma_1 u_b]\,[\bar{u}_b \Gamma_2 u_a]$$

- sum over the spin of b, use the completeness relation

$$\sum_{s_b} [\bar{u}_a \Gamma_2 u_b \bar{u}_b \bar{\Gamma}_2 u_a] \;\Rightarrow \bar{u}_a \Gamma_2 (\not{p}_b + m_b c) \bar{\Gamma}_2 u_a]$$

$$\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\mu p_\mu + mc)$$

- to sum over the spin of a, write out with indices

$$\sum_i \sum_j [\bar{u}_a]_i \; [\Gamma_2 (\not{p}_b + m_b c) \bar{\Gamma}_2]_{ij} \; [u_a]_j$$

- rearrange and introduce spin summation on a

$$\sum_i \sum_j \; [\Gamma_2 (\not{p}_b + m_b c) \bar{\Gamma}_2]_{ij} \; \sum_{s_a} [u_a]_j [\bar{u}_a]_i$$

$$[\not{p}_a + m_a c]_{ji}$$

SUM OVER SPINS OF FINAL STATES

$$\sum_i \boxed{\sum_j = \left[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2\right]_{ij}} \boxed{\left[\not{p}_a + m_a c\right]_{ji}}$$

A_{ij} B_{ji}

sum over j multiplies the two matrices

$[AB]_{ii}$

$$\sum_i \ [\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]_{ii}$$

$$\Rightarrow \ \mathrm{Tr}[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]$$

$$\sum_{s_a, s_b} [\bar{u}_a \Gamma_1 u_b] \left[\bar{u}_a \bar{\Gamma}_2 u_b\right]^* \Rightarrow \mathrm{Tr}\left[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)\right]$$

- If particle a or b are antiparticles (“v” spinors), reverse the sign of the mass term

TRACE RELATIONS:

Tr(A+B) = Tr A + Tr B
Tr(αA) = αTr A
Tr(AB) = Tr (BA)

- Tr[1] = 4

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu)$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \mathrm{Tr}(2g^{\mu\nu})$$

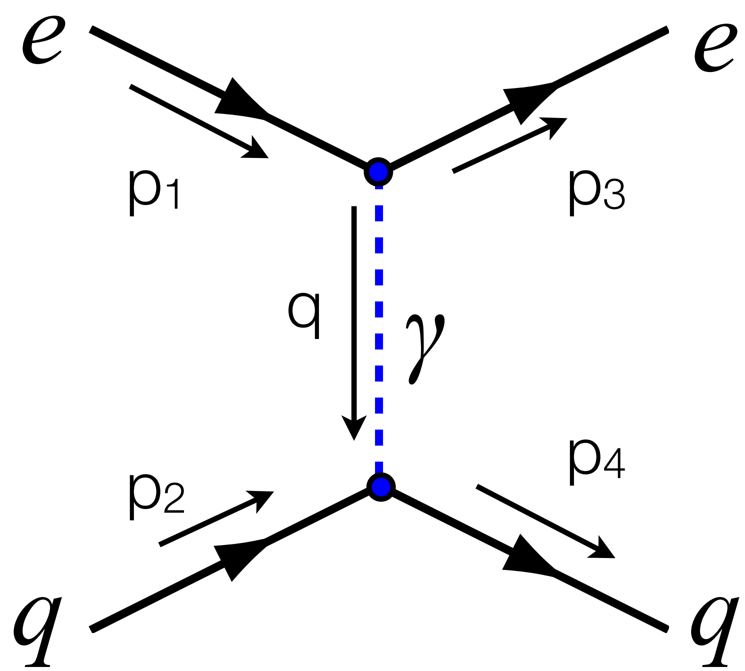
$$2 \times \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2 \times \mathrm{Tr}(g^{\mu\nu})$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4 \times g^{\mu\nu}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4 \times (g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$
- From here, we have:
  - $\mathrm{Tr}(\not a\ \not b) = 4a \cdot b$
  - $\mathrm{Tr}(\not a\ \not b\ \not c\ \not d) = 4\left[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\right]$
- Trace of any single γ matrix is 0
  - $\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5) = -\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu)$
  - Likewise, trace of the product of an odd number of γ matrices is 0

ELECTRON-QUARK SCATTERING



$$\bar{u}_3 \; (-ie\gamma^\mu) \; u_1 \; (2\pi)^4 \delta^4(p_1 - p_3 - q)$$

$$\bar{u}_4 \; (-iQe\gamma^\nu) \; u_2 \; (2\pi)^4 \delta^4(p_2 - p_4 + q)$$

$$\int \frac{dq^4}{(2\pi)^4} \frac{ig_{\mu\nu}}{q^2}$$

$$\mathcal{M} = \frac{Qe^2}{(p_1 - p_3)^2} \left[\bar{u}_3 \gamma^\mu u_1\right] \left[\bar{u}_4 \gamma_\mu u_2\right]$$

$$|\mathcal{M}|^2 = \frac{Q^2 e^4}{(p_1 - p_3)^4} \left[\bar{u}_3 \gamma^\mu u_1\right] \left[\bar{u}_3 \gamma^\nu u_1\right]^* \left[\bar{u}_4 \gamma_\mu u_2\right] \left[\bar{u}_4 \gamma_\nu u_2\right]^*$$

- summing over the sums of all the particles

$$\sum_{s_1, s_3} \left[\bar{u}_3 \gamma^\mu u_1\right] \left[\bar{u}_3 \gamma^\nu u_1\right]^* \Rightarrow \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + mc) \gamma^\nu (\not{p}_3 + mc)\right]$$

$$\sum_{s_a, s_b} \left[\bar{u}_a \Gamma_1 u_b\right] \left[\bar{u}_a \bar{\Gamma}_2 u_b\right]^* \Rightarrow \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2 (\not{p}_a + m_a c)\right]$$

$$\sum_{s_2, s_4} \left[\bar{u}_4 \gamma^\mu u_2\right] \left[\bar{u}_4 \gamma^\nu u_2\right]^* \Rightarrow \mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + Mc) \gamma_\nu (\not{p}_4 + Mtc)\right]$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{(p_1 - p_3)^4} \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + mc) \gamma^\nu (\not{p}_3 + mc)\right] \mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + Mc) \gamma_\nu (\not{p}_4 + Mc)\right]$$

EVALUATING TRACES

- Here are the trace expressions:

$$\mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)\right] = \mathrm{Tr}\left[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3\right] + m^2c^2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$

$$\mathrm{Tr}\left[\gamma_\mu(\not{p}_2 + Mc)\gamma_\nu(\not{p}_4 + Mc)\right] = \mathrm{Tr}\left[\gamma_\mu\not{p}_2\gamma_\nu\not{p}_4\right] + M^2c^2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$

- the relevant trace relations

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4 \times (g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4 \times g^{\mu\nu}$$

- we obtain the following:

$$\mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)\right] = 4\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1 \cdot p_3)g^{\mu\nu} + m^2c^2g^{\mu\nu}\right]$$

$$\mathrm{Tr}\left[\gamma_\mu(\not{p}_2 + Mc)\gamma^\nu(\not{p}_4 + Mc)\right] = 4\left[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - (p_2 \cdot p_4)g_{\mu\nu} + M^2c^2g^{\mu\nu}\right]$$

ENDGAME

- Contract the two trace expressions

$$4\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1 \cdot p_3)g^{\mu\nu} + m^2c^2 g^{\mu\nu}\right]$$

$$4\left[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - (p_2 \cdot p_4)g_{\mu\nu} + M^2c^2 g_{\mu\nu}\right]$$

$$\Rightarrow 32\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (M^2c^2)(p_1 \cdot p_3) - m^2c^2(p_2 \cdot p_4) + 2m^2c^2M^2c^2\right]$$

- Putting it back into the amplitude expression

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{32Q^2e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (M^2c^2)(p_1 \cdot p_3) - m^2c^2(p_2 \cdot p_4) + 2m^2c^2M^2c^2\right]$$

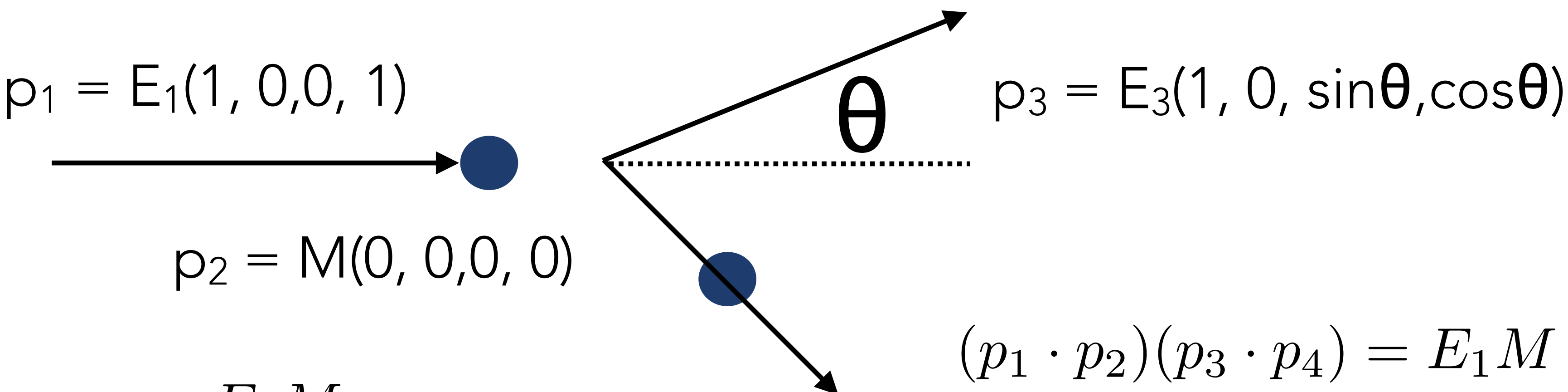
- don’t forget we need 1/4 averaging factor
- Put it into the phase space expression for lab-frame scattering (note this expression ignores the electron mass)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |\mathcal{M}|^2$$

KINEMATICS:

- First let's drop the electron mass

$$\langle|\mathcal{M}|^2\rangle = \frac{8Q^2e^4}{(p_1 - p_3)^4}\left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (M^2c^2)(p_1 \cdot p_3)\right]$$



$$(p_1 \cdot p_2)(p_3 \cdot p_4) = E_1 M \times (E_1 E_3(1 - \cos\theta) + E_3 M)$$

$$(p_1 \cdot p_4)(p_2 \cdot p_3) = E_3 M \times (E_1 M - E_1 E_3(1 - \cos\theta))$$

$$M^2c^2(p_1 \cdot p_3) = M^2 E_1 E_3 (1 - \cos\theta)$$

$$p_1 \cdot p_2 = E_1 M$$

$$p_2 \cdot p_3 = E_3 M$$

$$p_1 \cdot p_3 = E_1 E_3 (1 - \cos\theta)$$

$$p_1 \cdot p_4 = p_1 \cdot (p_1 + p_2 - p_3) \sim p_1 \cdot p_2 - p_1 \cdot p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$p_3 \cdot p_4 = p_3 \cdot (p_1 + p_2 - p_3) \sim p_1 \cdot p_3 - p_2 \cdot p_3 = E_1 E_3 (1 - \cos\theta) + E_3 M$$

$$\langle|\mathcal{M}|^2\rangle = \frac{8Q^2e^4}{(p_1 - p_3)^4} E_1 E_3 M \left[(E_1 - E_3)(1 - \cos\theta) + M(1 + \cos\theta)\right]$$

$$= \frac{16Q^2e^4}{(p_1 - p_3)^4} E_1 E_3 M \left[(E_1 - E_3)\sin^2\frac{\theta}{2} + M\cos^2\frac{\theta}{2}\right]$$

MOMENTUM TRANSFER:

$$\langle|\mathcal{M}|^2\rangle = \frac{16Q^2e^4}{(p_1-p_3)^4}E_1E_3M\left[(E_1-E_3)\sin^2\frac{\theta}{2}+M\cos^2\frac{\theta}{2}\right]$$

$$(p_1-p_3)^2 \sim -2p_1\cdot p_3 = -4E_1E_3\sin^2\frac{\theta}{2} = q^2 \equiv -Q^2$$

$$Q^2 = 2M(E_1-E_3)$$

$$\langle|\mathcal{M}|^2\rangle = \frac{M^2e^4}{E_1E_3\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}+\frac{Q^2}{2M}\sin^2\frac{\theta}{2}\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|\mathcal{M}|^2$$

$$e^2 = 4\pi\alpha$$

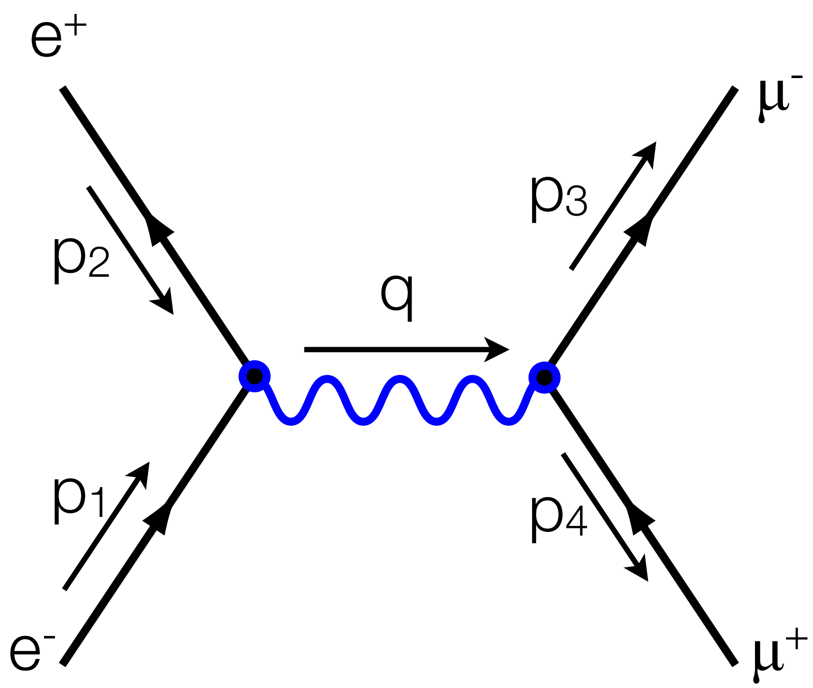
$$= \frac{e^4}{64\pi^2}\frac{1}{E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\left[\cos^2\frac{\theta}{2}+\frac{Q^2}{2M}\sin^2\frac{\theta}{2}\right] = \frac{\alpha^2}{4}\frac{1}{E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\left[\cos^2\frac{\theta}{2}+\frac{Q^2}{2M}\sin^2\frac{\theta}{2}\right]$$

A FEW NOTES

- Note that there is only one degree of freedom in the scattering:
  - $$E_3 = \frac{E_1 M}{M + E_1(1 - \cos\theta)}$$
  - $$Q^2 = \frac{2ME_1^2(1 - \cos\theta}{M + E_1(1 - \cos\theta)}$$
  - i.e. if we know one of E3, Q, or Q2, the others are determined.

NEXT TIME:

- Please read
  - 7.5, 9.1
  - if you have a chance, 10.1



$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\;\gamma^\mu\;v(4)]\,[\bar{v}(2)\;\gamma_\mu\;u(1)]$$

$$\mathcal{M}^* = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\gamma^\nu v(4)]^*\,[\bar{v}(2)\gamma_\nu u(1)]^*$$

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4}\,[\bar{u}(3)\gamma^\mu v(4)]\,[\bar{u}(3)\gamma^\nu v(4)]^*\,[\bar{v}(2)\gamma_\mu u(1)]\,[\bar{v}(2)\gamma_\nu u(1)]^*$$

$$\sum_{\text{spins}}\,[\bar{u}(3)\gamma^\mu v(4)]\,[\bar{u}(3)\gamma^\nu v(4)]^* = \mathrm{Tr}\,[(\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)]$$

$$\sum_{\text{spins}}\,[\bar{v}(2)\gamma^\mu u(1)]\,[\bar{v}(2)\gamma^\nu u(1)]^* = \mathrm{Tr}\,[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)]$$

$$\sum_{\text{spins}} [\bar{u}(3)\gamma^\mu v(4)] [\bar{u}(3)\gamma^\nu v(4)]^* = \mathrm{Tr}\,[(\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)]$$

$$\sum_{\text{spins}} [\bar{v}(2)\gamma^\mu u(1)] [\bar{v}(2)\gamma^\nu u(1)]^* = \mathrm{Tr}\,[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)]$$

$$\mathrm{Tr}\left[\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)\right] = 4\left(p_4^\mu p_3^\nu + p_4^\nu p_3^\mu - g^{\mu\nu}(p_3 \cdot p_4) - 4Mc^2 g^{\mu\nu}\right)$$

$$\mathrm{Tr}\left[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)\right] = 4\left(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - g_{\mu\nu}(p_1 \cdot p_2) - 4mc^2 g_{\mu\nu}\right)$$