Decay Rates, Cross Sections and Phase Space

H. A. Tanaka

Overview

- Decay rates and lifetimes
  - relation to half-life, total decay rates, etc.
- Fermi's golden rule
  - how phase space affects interactions/transitions/reactions/decays
- How to calculate phase space
  - general formula for the decay of a particle
  - computation of total rate of 2-body decay

Particle Decays:

- A particle of a given type is identical to all others of its type
  - some probability to decay within an infinitesimal time period dt (call it Γ)
  - Γ is independent of how “old” the particle is.
- If we have an ensemble of these particles, the total rate of change of the number of particles is given by:

$$dN = -\Gamma \, N dt \quad \Rightarrow \quad N(t) = N_0 e^{-\Gamma t} \qquad N_0 \equiv N(0)$$

- The number of surviving particles follows an exponential distribution
  - if we wait for half of the particles to disappear, we can define “half life”

$$\frac{N(t)}{N_0} = \frac{1}{2} = e^{-\Gamma t} \quad \Rightarrow \quad t_{1/2} = \frac{\log 2}{\Gamma}$$

  - if we wait for the number to decrease by a factor of e, we define “lifetime”

$$\frac{N(t)}{N_0} = \frac{1}{e} = e^{-\Gamma t} \quad \Rightarrow \quad \tau = \frac{1}{\Gamma}$$

Combining Decay Rates:

- If a particle has several decay "modes" each with a given rate Γ_i, the total decay rate is given by the sum of all the rates:

$$\Gamma_{tot} = \sum_i \Gamma_i \quad \Rightarrow \quad \tau = \frac{1}{\Gamma_{tot}}$$

- If you are observing only one of these decay modes as a function of time, you will still see the number of particles diminish as the total decay rate

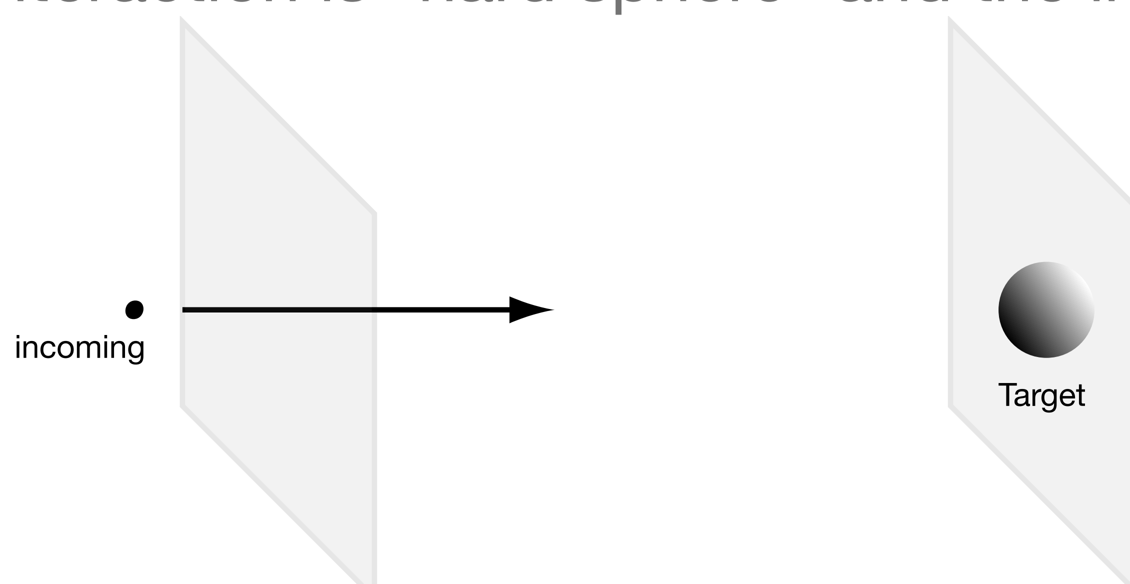
$$e^{-\Gamma_{tot} t} = e^{-t/\tau}$$

even though the rate of decay per unit time is a fraction of the total decay rate

  - You are observing a fraction of the total decays which means that the distribution will diminish as that fraction times the overall exponential.

Scattering Rates:

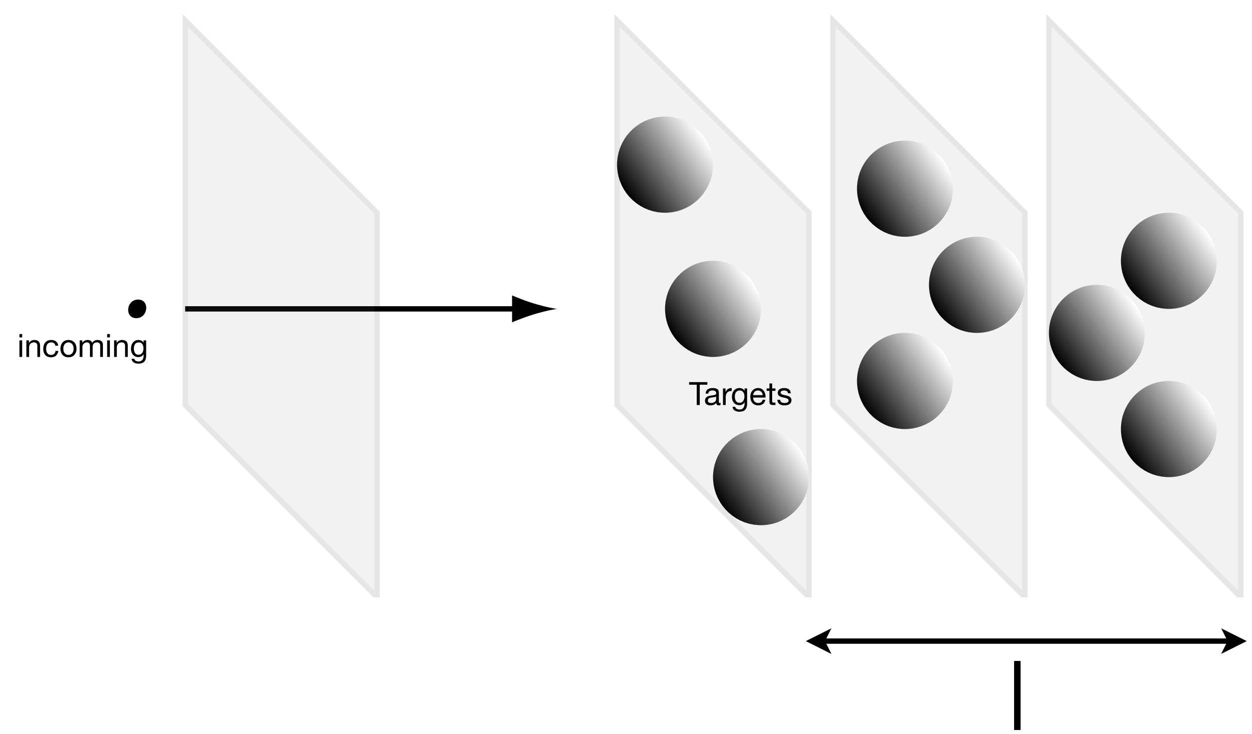
- Scattering
  - send in particles on a “target” and study what comes out
  - total rate corresponds to the rate at which “something” happens.
- Consider sending in one particle in a unit area occupied by one target particle, assuming the interaction is “hard sphere” and the incoming particle is infinitesimal:



- Probability of interaction is area of target/unit area: area = “cross section” σ
- Rate $\propto$ rate of incoming particles: Luminosity $\mathcal{L}$ = particles/unit area/time

More than one target:

- We can consider the possibility that there is more than one “layer” of target particles, or that there is more than one target per unit area.



- The rate is scaled by the number of targets in the column swept by the incoming beam
  - Rate = N_T/Unit Area x σ x $\mathcal{L}$ = n l σ $\mathcal{L}$
  - n = number density of target particles, l = length of target area.

Generalizing the idea of a cross section:

- In hard sphere scattering, something “happening” is binary:
  - If the balls hit each other, then something happened
  - otherwise, nothing happened
- We generalize the idea of “something happening” by considering “differential cross section.”
  - Probability that some particle ends up in a particular part of phase space
  - e.g.. a particular momentum/angle range.

polar angle

$$\sigma \Rightarrow \frac{d^3\sigma}{d\Omega\ dp} \qquad d\Omega = \sin\theta d\theta\ d\phi = d\cos\theta\ d\phi$$

“solid angle” azimuthal angle

  - The notation lends itself to “integrating” over a phase space variable: say we don’t care about the momentum but only the angle:

$$\frac{d\sigma}{d\Omega} = \int p^2 dp \frac{d^3\sigma}{d\Omega\ dp}$$

Total Cross section

- Likewise, we can integrate over all phase space to get the total cross section:

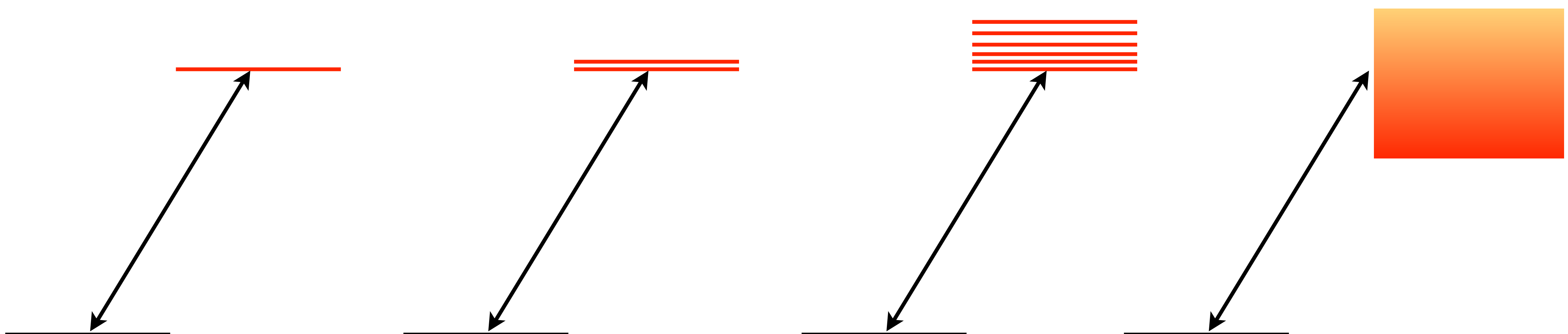
$$\sigma_{TOT} = \int p^2 dp\, d\phi\, d\cos\theta \frac{d^3\sigma}{d\Omega\, dp}$$

- This is the cross section for a particle to end up anywhere in phase space
- Note for “infinite range” interactions like the Coulomb interaction, the total cross section can be infinite; i.e. “something” always happens
  - This just reflects the fact that no matter how far you are away, there is still some electric field that will deflect your particle.

Golden Rule:

- Fermi's golden rule states that the probability of a transition in quantum mechanics is given by the product of:
    - The absolute value of the matrix element (a k a amplitude) squared
    - The available density of states.

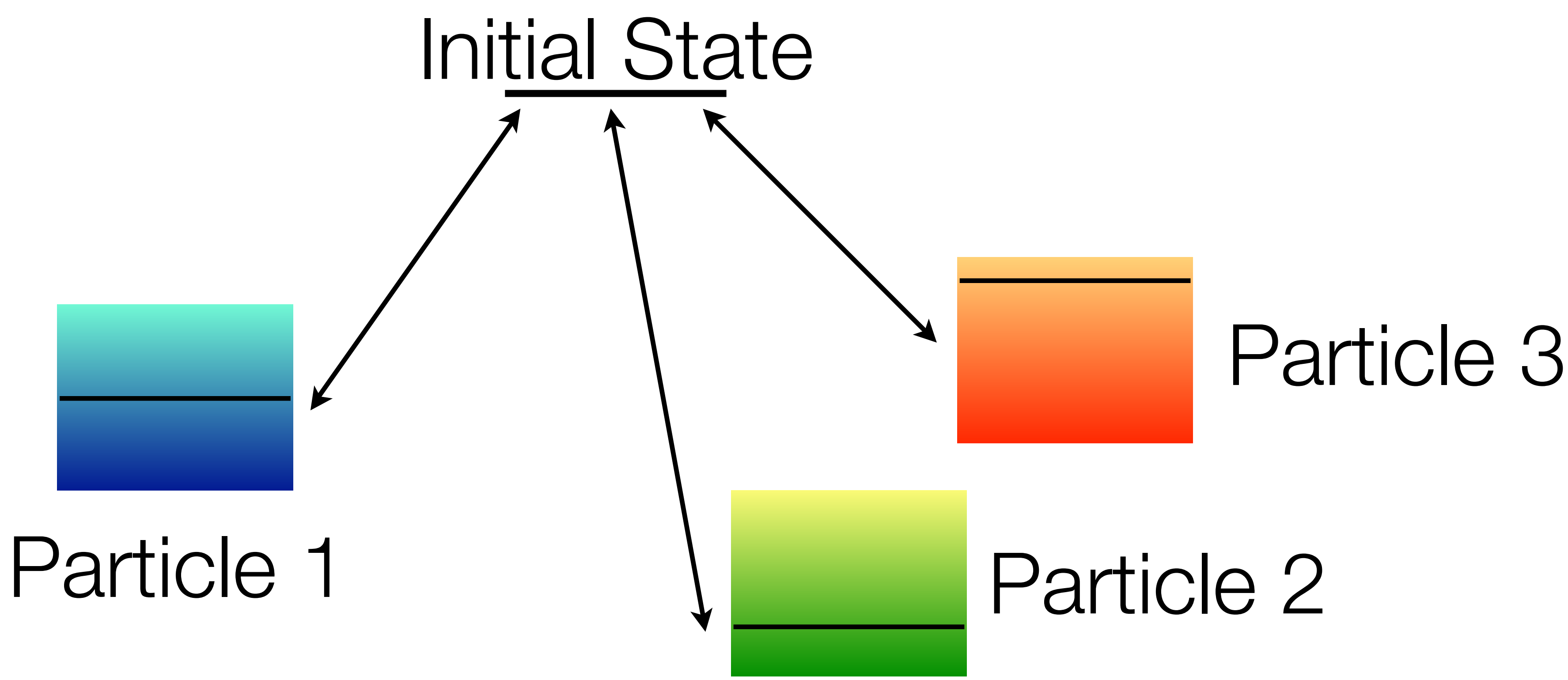
$$P \propto |\mathcal{M}|^2 \times \rho$$



$$P \propto |\mathcal{M}|^2 \qquad P \propto 2 \times |\mathcal{M}|^2 \qquad P \propto \int |\mathcal{M}(E)|^2 \, \rho(E) \, dE$$

- Typically a decay of a particle into states with lighter product masses has more "phase space" and more likely to occur.
- Now let's see how to calculate the phase space (deal with amplitude later)

Product of Phase space



- What is net phase space for the particle 1,2,3 to end up in particular places?
  - 0 if energy and momentum are not conserved
  - 0 if particles are not on "mass shell"
- Otherwise, the product of the individual phase spaces:

$$\rho = \rho_1(p_1^\mu) \times \rho_2(p_2^\mu) \times \rho_3(p_3^\mu)$$

integral extends over region satisfying kinematic constraints

$$\rho_{tot} = \int_{allowed} \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\ \rho_1(p_1^\mu)\rho_2(p_2^\mu)\rho_3(p_3^\mu)$$

The Dirac δ and Heaviside-Lorentz Θ

- δ(x)
  - is 0 for all x≠0
  - spikes at x=0 such that $\int_{-\infty}^{\infty} dx\ \delta(x) = 1$
- We can deduce that:
  - $\int_{-\infty}^{\infty} dx\ f(x)\delta(x) = f(0)$ $\quad$ $\delta(f(x)) = \sum_{x_i | f(x_i)=0} \frac{1}{|f'(x_i)|}\delta(x - x_i)$
- We can also have multi-dimensional δ functions

$$\delta(\vec{x}) \qquad \delta(x^\mu)$$

$$\delta(\vec{x}) = \delta(x)\delta(y)\delta(z) \qquad \delta(x^\mu) = \delta(x^0)\delta(x^1)\delta(x^2)\delta(x^3) = \delta(x^0)\delta(\vec{x})$$

- Closely related is the Θ(x) (“Heavyside-Lorentz”) function:
  - 0 for x<0, 1 for x>1

Use these to enforce energy/momentum conservation and masses

Phase Space in Decays

$$\Gamma = \frac{S}{2\hbar m} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \; \delta^4\Big(\sum_i p_i^\mu - \sum_f p_f^\mu\Big) \times \prod_{j=2}^{N} 2\pi \; \delta(p_j^2 - m_j^2 c^2) \; \Theta(p_j^0) \; \frac{d^4 p_j}{(2\pi)^4}$$

Symmetry factor

Matrix element factor (function of kinematics, polarizations, etc.)

Energy and momentum must be conserved

Product over all outgoing particles

Outgoing particles must be on mass shell

Energy must be positive

distributed evenly in phase space

- Complicating looking, but represents a basic statement:
  - apart from matrix element, phase space is distributed evenly among all particles subject to mass requirements, E/p conservation
  - “dynamics” like parity violation, etc. incorporated into matrix element.

The Symmetry Factor:

$$\int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4}$$

- Consider the integration of phase space for two particles in the final state where the particles are of the same species.
- At some point, say, there will be a configuration where $p_1 = K_1$ and $p_2 = K_2$
  - Since the particles are identical, we should also have the reverse case:
    - $p_1 = K_2$, $p_2 = K_1$
    - the integral will contain both cases separately.
  - However, in quantum mechanics, the identicalness of particles of the same species means that these are the same state and we have double counted.
  - We need to add a factor of 1/2 to the phase space
- Likewise, for n identical particles in the final state, we need a factor of 1/n!

Working with the Golden Rule: 2-body decay

$$\Gamma \quad = \quad \frac{S}{2\hbar m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \; \delta^4(p_1^\mu - p_2^\mu - p_3^\mu)$$

$$\times \delta(p_2^2 - m_2^2 c^2)\Theta(p_2^0) \;\; \times \;\; \delta(p_3^2 - m_3^2 c^2)\Theta(p_3^0)$$

$$\frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4}$$

Let's integrate over overall outgoing particle phase space to get the total decay rate

- Start with the phase space factors: $d^4 p \equiv dp^0 dp^1 dp^2 dp^3$

$$\delta(p^2 - m^2 c^2) = \delta((p^0)^2 - \vec{p}^2 - m^2 c^2)$$

$$\delta(p^2 - m^2 c^2) \quad = \frac{1}{2 \times \sqrt{\vec{p}^{\,2} + m^2 c^2}} \left[\delta(p^0 - \sqrt{\vec{p}^{\,2} + m^2 c^2}) + \delta(p^0 + \sqrt{\vec{p}^{\,2} + m^2 c^2}) \right]$$

- Now consider the fact that we have $\Theta(p^0)$: we can eliminate the second delta function since $\Theta(p^0)$ will be zero whenever p^0 is negative

$$\delta(p^2 - m^2 c^2) \quad = \frac{1}{2 \times \sqrt{\vec{p}^{\,2} + m^2 c^2}} \delta(p^0 - \sqrt{\vec{p}^{\,2} + m^2 c^2})$$

$$p_0 \Rightarrow \sqrt{\vec{p}^{\,2} + m^2 c^2}$$

Enforcing Energy/Momentum Conservation

- Now integrate over p^0_3 and p^0_2 using the previous relations

$$\Gamma = \frac{S}{2\hbar m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4\ \delta^4(p_1^\mu - p_2^\mu - p_3^\mu)$$

$$\times \underbrace{2\pi \times \delta(p_2^2 - m_2^2c^2)\Theta(p_2^0)}_{\downarrow\ \frac{1}{2\times\sqrt{\vec{p_2}^{\ 2} + m_2^2c^2}}} \times \underbrace{2\pi \times \delta(p_3^2 - m_3^2c^2)\Theta(p_3^0)}_{\downarrow\ \frac{1}{2\times\sqrt{\vec{p_3}^{\ 2} + m_3^2c^2}}}$$

$$\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{p}_3}{(2\pi)^3}\cancel{\frac{dp_2^0}{(2\pi)}}\cancel{\frac{dp_3^0}{(2\pi)}}$$

note p^0_2 and p^0_3 are now set according to E/p conservation by the δ function

$$\Gamma = \frac{S}{32\pi^2\hbar m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\vec{p_2}^{\ 2} + m_2^2c^2}\ \sqrt{\vec{p_3}^{\ 2} + m_3^2c^2}}\ d^3\vec{p}_2\ d^3\vec{p}_3$$

Decay at Rest:

- Decompose the product delta function (particle 1 at rest)

$$\delta^4(p_1^\mu - p_2^\mu - p_3^\mu) \Rightarrow \delta(m_1 c - \sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2} - \sqrt{\vec{p}_3^{\ 2} + m_3^2})\ \delta^3(-\vec{p}_2 - \vec{p}_3)$$

- Perform the d^3p_3 integral

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta^4(p_1^\mu - p_2^\mu - p_3^\mu)}{\sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2}\ \sqrt{\vec{p}_3^{\ 2} + m_3^2 c^2}}\ d^3\vec{p}_2\ \cancel{d^3\vec{p}_3}$$

$$\Big\downarrow$$

$$\sqrt{\vec{p}_2^{\ 2} + m_3^2 c^2}$$

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \times \int |\mathcal{M}|^2 \times \frac{\delta(m_1 c - \sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2} - \sqrt{\vec{p}_2^{\ 2} + m_3^2 c^2})}{\sqrt{\vec{p}_2^{\ 2} + m_2^2 c^2}\ \sqrt{\vec{p}_2^{\ 2} + m_3^2 c^2}}\ d^3\vec{p}_2$$

Integral in spherical coordinates

- $d^3\vec{p}_2 \Rightarrow d\phi\, d\cos\theta\, |p_2|^2 dp_2$

Assume no dependence of M on p

$$\Gamma = \frac{S}{32\pi^2\hbar m_1} \times \int d\phi\, d\cos\theta\, |\mathbf{p}_2|^2\, d|\mathbf{p}_2||\mathcal{M}|^2 \times \frac{\delta(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2c^2})}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\,\sqrt{\mathbf{p}_2^2 + m_3^2c^2}}$$

$$\int_0^{2\pi} d\phi \to 2\pi \qquad \int_{-1}^{+1} d\cos\theta \to 2$$

$$\begin{aligned} u &= \sqrt{\mathbf{p}_2^2 + m_2^2c^2} + \sqrt{\mathbf{p}_2^2 + m_3^2c^2} \\ du &= \frac{u|\mathbf{p}_2|}{\sqrt{\mathbf{p}_2^2+m_2^2c^2}\sqrt{\mathbf{p}_2^2+m_3^2c^2}} d|\mathbf{p}_2| \end{aligned}$$

do a change of variables to enact the final integral

$$\Gamma = \frac{S}{8\pi\hbar m_1} \times \int_{p_2=0,u=m_2+m_3}^{\infty} du\, |\mathcal{M}|^2 \times\ \delta(m_1 c - u)\frac{|\mathbf{p}_2|}{u}$$

The final integral over u sends $u=m_1c$ and makes p_2 consistent with E conservation

Final Result: Total two-body decay rate:

$$\Gamma = \frac{S}{8\pi\hbar m_1^2 c} \times |\mathcal{M}|^2 \times |\vec{p}_2|$$

why $|\mathbf{p}_2|$ and not $|\mathbf{p}_3|$?

- We now need to be able to calculate the matrix element M
  - Thus far (in isospin, etc.) we have dealt with relationships between different M based on symmetry rather than calculating M directly
  - later, we'll use Feynman diagrams and the associated calculus to calculate amplitudes for various elementary processes

Summary

- Decay rates: constant probability of a particle to decay in any interval results in exponential lifetime distribution
  - exponential is governed by one overall lifetime τ
  - τ is given by the inverse of the sum of the decay rates for all channels
- Phase space:
  - formalized the idea of what “phase space” is based on Fermi’s golden rule
  - it is the volume of the total allowed kinematic configurations assuming
    - outgoing particles are “on mass shell” i.e. have their nominal mass
    - energy and momentum are conserved
    - energy of each particle is positive
    - every allowed kinematic configuration has equal “weight”
- Starting with a general “differential” decay rate or scattering cross section, integrate over some of the outgoing quantities to get the dependence on the quantities or all of them to get the total rate