Lecture 6: Symmetries in Quantum Mechanics

H. A. Tanaka

Outline

- Introduce symmetries and their relevance to physics in general
    - Examine some simple examples
    - Consequences of symmetries for physics
    - groups and matrices
- Study angular momentum as a symmetry group
    - commutation relations
    - simultaneous eigenstates of angular momentum
    - adding angular momentum: highest weight decomposition
    - Clebsch-Gordan coefficients:
        - what are they?
        - how to calculate or find their values.
- Warning: this may be a particularly dense lecture (fundamental QM ideas)
    - **You should have seen this before in quantum mechanics**

Symmetries

- symmetry: an operation (on something) that leaves it unchanged
  - rotations/reflections: triangles (isosceles, equilateral), square, rectangle
  - translation: (crystal lattice)
  - discrete/continuous: rotations of a square vs. a circle
- Mathematically, symmetries form mathematical objects called groups:
  - closure: 2 symmetry operations make another one (member of the group)
  - identity: doing nothing is a symmetry operation (member of the group)
  - inverse: for each operation, there is another one that undoes it (i.e. operation + inverse is equivalent to the identity)
  - Associativity: $R_1(R_2R_3)= (R_1R_2)R_3$
- Nöther's Theorem:
  - Symmetry in a system corresponds to a conservation law
    - Space and translation symmetry → conservation of ?
    - Rotational symmetry → conservation of ?

Matrices

- Symmetry operations can often be expressed (“represented”) by matrices
  - The composition of two operations translates into matrix multiplication
  - How do group properties (closure, identity, inverse, associativity) translate?
- Some important groups of matrices in physics:
  - U(N): N x N unitary matrices, $U^{-1}=U^{T*}$
  - SU(N): U(N) matrices with determinant 1
  - O(N): N x N orthogonal matrices: real matrices with $O^{-1}=O^{T}$
  - SO(N): O(N) matrices with determinant 1
- The rotation group and angular momentum are fundamentally associated with the group SU(2) (~SO(3)) and its representations.
  - “2x2 unitary matrices with determinant 1”
  - Groups have larger dimension representations (i.e. NxN matrices) with the same structure. In SU(2), these correspond to systems with different total angular momentum.

Commutation Relations:

- Matrix multiplication is order-dependent!
  - Generally: A B ≠ B A
  - Call AB -BA the commutator of A, B
    - [A, B] = A B - B A

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Quantum Mechanics:
  - Observable quantities correspond to operators (matrices), eigenvalues
  - A state with a well-defined value for an observable is an eigenvector of the corresponding operator. The value is the eigenvalue for the eigenvector.
  - Eigenvectors with different eigenvalues are orthogonal
  - For two observables, if the order of the measurement matters, then a state cannot simultaneously be an eigenvector of both operators
- Example:
  - $[x, p_x] = i\hbar$: x, p_x cannot simultaneously have well-defined values
  - More generally, we can say $[x_i, p_j] = \delta_{ij}\, i\hbar$

Playing around with the Commutator

- At home, you should convince yourself of the following:
  - $[A,B] = -[B,A]$
  - $[A, B+C] = [A,B] + [A,C]$
  - $[A+B, C] = [A,C] + [B,C]$
  - $[A,A] = 0$
  - $[A,BC] = [A, B] C + B [A,C]$
  - $[AB, C] = A [B,C] + [A, C] B$
  - $[A, A^2] = [A, A^n] = 0$

Commutation Relations for Angular Momentum

- We can express angular momentum in terms of its classical counterparts and introduce a new notation:

  $$\vec{L} = \vec{r} \times \vec{p} \rightarrow L_i = \epsilon_{ijk} x_j p_k$$

  where the "completely antisymmetric tensor" ε_{ijk} is defined by:
  - $\varepsilon_{ijk} = 1$ if i, j, k are an even permutation of 1, 2, 3 = (x,y,z)
  - $\varepsilon_{ijk} = -1$ if i, j, k are odd permutation of (x, y, z)
  - $\varepsilon_{ijk} = 0$ otherwise (if any of i, j, k are the same)
- Examine $[L_x, L_y] = [y\, p_z - z\, p_y, z\, p_x - x\, p_z]$

  pay attention to what commutes and what doesn't

  - $(y\, p_z - z\, p_y)(z\, p_x - x\, p_z) = y\, p_z\, z\, p_x - y\, p_z\, x\, p_z - z\, p_y\, z\, p_x + z\, p_y\, x\, p_z$
  - $(z\, p_x - x\, p_z)(y\, p_z - z\, p_y) = z\, p_x\, y\, p_z - x\, p_z\, y\, p_z - z\, p_x\, z\, p_y + x\, p_z\, z\, p_y$
    - $= -p_x\, y\, [z, p_z] + x\, p_y\, [z, p_z] = i\hbar\, (x\, p_y - y\, p_x) = i\hbar\, L_z$
  - More generally, we can write $[L_i, L_j] = i\hbar\, \varepsilon_{ijk}\, L_k$
- States cannot be eigenvectors of more than one L_i

Total Angular Momentum:

- Define the operator $L^2 = L_x^2 + L_y^2 + L_z^2$
  - (Aside: $[AB, C] = A [B,C] + [A,C] B$)
- Then the commutator $[L^2, L_x] = 0$
  - $[L_x^2, L_x] = 0$
  - $[L_y^2, L_x] = L_y [L_y, L_x] + [L_y, L_x] L_y = -i\hbar(L_y L_z + L_z L_y)$
  - $[L_z^2, L_x] = L_z [L_z, L_x] + [L_z, L_x] L_z = i\hbar(L_z L_y + L_y L_z)$
- States can simultaneously be eigenvectors of total angular momentum and one component of angular momentum
- Conventionally, this direction is taken as z:

$$\begin{aligned} |l,m\rangle \quad &\rightarrow \quad L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle \\ &\rightarrow \quad L_z|l,m\rangle = \hbar m|l,m\rangle \end{aligned}$$

- For *orbital* angular momentum, l must be a (positive) integer
- For *spin* angular momentum, l can be half or whole integer

The “Ladder” Operator:

- Consider the operator $L_+ = L_x + iL_y$, in particular its commutator
  - $[L_z, L_+] = [L_z, L_x + iL_y] = [L_z, L_x] + i[L_z, L_y] = i\hbar L_y + \hbar L_x = \hbar L_+$
- Now consider: $L_z(L_+|l,m\rangle)$
  - Using the commutation relation we just derived

$$\begin{aligned} L_z L_+|l,m\rangle &= (L_+L_z + \hbar L_+)|l,m\rangle \\ &= \hbar(m+1)L_+|l,m\rangle \end{aligned}$$

  - $L_+|l,m\rangle$ is an eigenstate of L_z with eigenvalue $\hbar(m+1)$
- Likewise, with $L_- = L_x - iL_y$, we can show $L_-|l,m\rangle$ is an eigenstate of L_z with eigenvalue $\hbar(m-1)$
- L_+ and L_- are called “ladder” or “raising and lowering” operators

Normalizing the State

- Consider the inner product of a state with itself: $\langle\alpha|\alpha\rangle$
  - $|\langle\beta|\alpha\rangle|^2$ is the probability that a state $|\alpha\rangle$ can be found in the state $|\beta\rangle$
  - For (possibly) obvious reasons, we want $\langle\alpha|\alpha\rangle = 1$
    - we then say that the state is normalized
    - we have assumed thus far that our angular momentum states $|l, m\rangle$ are normalized, i.e. $\langle l, m|l, m\rangle = 1$
  - In general, the normalization of a state resulting from an operation can change: $\langle\alpha|O^\dagger O|\alpha\rangle \neq 1$
    - we need to "renormalize" the state by rescaling it.
- For states produced by the ladder operators, we obtain the normalization by calculating $\langle l, m|L_+^\dagger L_+|l, m\rangle = \langle l, m|L_- L_+|l, m\rangle$
- If we consider $L_- L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + i(L_x L_y - L_y L_x)$
  - $= L_x^2 + L_y^2 + i\, h\, i\, L_z = L^2 - L_z^2 - \hbar L_z$

Check:

- Is everyone happy with :

$$|\frac{1}{2},\frac{1}{2}\rangle \qquad |2,-1\rangle$$

m = z-axis component

l=total angular momentum number

- What about:

$$|\frac{1}{2},\frac{1}{2}\rangle|2,-1\rangle$$

  - two objects, one spin 1/2, the other l=2.
  - First object has s_z=1/2, second is l_z=-1

Climbing up and down the ladder

- Now inserting the operator back into the equation and recall the fact that the state is an eigenvalue of L^2 and L_z

  $$\langle l,m|L^2 - L_z^2 - \hbar L_z|l,m\rangle = \langle l,m|\hbar^2 l(l+1) - m^2\hbar^2 - m\hbar^2|l,m\rangle$$

- Thus, if we call $|l,m+1\rangle$ the normalized eigenvector with eigenvalues l(l+1) and m for L^2 and L_z, respectively, then

$$L_+|l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}|l,m+1\rangle$$

$$L_-|l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}|l,m-1\rangle$$

- Note:
  - If we act with L_+ on $|l,m=l\rangle$, we get zero
  - If we act with L_- on $|l,m=-l\rangle$, we get zero
- The “top” and “bottom” of the ladder are at $m = \pm\, l$
  - For a given l, m ranges from $-l$ to l in integer steps.

Adding Angular Momentum:

- We have two objects with angular momentum states and wish to consider the total angular momentum:
  - We have three sets of eigenstates:
    - The l, z eigenstates that we are adding together $|l_1, m_1\rangle, |l_2, m_2\rangle$
    - The l,z states of the summed state: $|J, J_z\rangle$
  - Recall that angular momentum is a (axial) vector quantity:
    - How do the two separate states correspond to the combined angular momentum states?
- The components of L_z (i.e. m_1, m_2) add:
  - $J_z = m_1 + m_2$
- The combined total angular momentum can have a range:
  - $J = |l_1 - l_2|$ to $|l_1 + l_2|$: corresponds to initial states anti-parallel or parallel.

Highest Weight Decomposition

- Consider the combination of two spin 1/2 objects: $s_{1,2}=1/2$.
  - There are four independent states: $|\frac{1}{2},\pm\frac{1}{2}\rangle|\frac{1}{2},\pm\frac{1}{2}\rangle$
    - These can add to form states of total angular momentum J = 0 or 1
    - There are four independent states (1 for J=0, 3 for J = 1)
- Consider the state $|J=1, J_z=1\rangle$
  - since there $J_z = m_1 + m_2$, this must correspond to the state $|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle$
  - What about the other states? We apply the "lowering operator" L_- on both sides of the equation. Note that $\vec{J}=\vec{s}_1+\vec{s}_2$ so $J_-=s_{1-}+s_{2-}$
- On the "combined side": $J_-|1,1\rangle=\sqrt{1\times 2-1\times 0}|1,0\rangle=\sqrt{2}|1,0\rangle$
- For the "individual" states:

$$(s_{1-}+s_{2-})|\tfrac{1}{2},\tfrac{1}{2}\rangle\;|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}\tfrac{3}{2}-\tfrac{1}{2}\tfrac{-1}{2}}|\tfrac{1}{2},\tfrac{-1}{2}\rangle\;|\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{1}{2}\tfrac{3}{2}-\tfrac{1}{2}\tfrac{-1}{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle\;|\tfrac{1}{2},\tfrac{-1}{2}\rangle$$
$$= |\tfrac{1}{2},\tfrac{-1}{2}\rangle\;|\tfrac{1}{2},\tfrac{1}{2}\rangle+|\tfrac{1}{2},\tfrac{1}{2}\rangle\;|\tfrac{1}{2},\tfrac{-1}{2}\rangle$$

$$|1,0\rangle=\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle+|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

To the bottom of the latter

- Now apply J_- and $s_{1-} + s_{2-}$ to $|1,0\rangle$ and $\frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle + |\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle\right)$ to obtain $|1,-1\rangle$

  Note: $s_{1-}|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle = 0$ $\quad s_{2-}|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle = 0$

- How do we obtain the J=0 state $|0,0\rangle$?
  - recall that states with different eigenvalues for an operator are orthogonal
  - Thus $|0,0\rangle$ must be orthogonal to $|1,0\rangle$ (i.e. J eigenvalues are different)
    - in the component space, it must be orthogonal to
      $\frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle + |\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle\right)$
    - The only available orthogonal state with the right quantum numbers is

$$\frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle - |\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle\right)$$

Determining the orthogonal state:

We concluded that: $|1,0\rangle = \frac{1}{\sqrt{2}}|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle$

and now we want to find $|0,0\rangle$

- How do we show two states $|A\rangle$ and $|B\rangle$ are orthogonal?
  - We consider $\langle A|B\rangle$ or $\langle B|A\rangle$: this should be 0
- We postulated that $\frac{1}{\sqrt{2}}|\frac{1}{2},\frac{-1}{2}\rangle\,|\frac{1}{2},\frac{+1}{2}\rangle - \frac{1}{\sqrt{2}}|\frac{1}{2},\frac{+1}{2}\rangle\,|\frac{1}{2},\frac{-1}{2}\rangle$ is orthogonal
- So we consider: $[\,\langle\frac{1}{2},\frac{-1}{2}|\langle\frac{1}{2},\frac{+1}{2}| - \langle\frac{1}{2},\frac{+1}{2}|\langle\frac{1}{2},\frac{-1}{2}|\,]\,[\,|\frac{1}{2},\frac{-1}{2}\rangle|\frac{1}{2},\frac{+1}{2}\rangle + |\frac{1}{2},\frac{+1}{2}\rangle|\frac{1}{2},\frac{-1}{2}\rangle\,]$
- Recalling that:

=0

$$\langle\frac{1}{2},\frac{-1}{2}|\langle\frac{1}{2},\frac{+1}{2}||\frac{1}{2},\frac{-1}{2}\rangle|\frac{1}{2},\frac{+1}{2}\rangle = 1$$

$$\langle\frac{1}{2},\frac{+1}{2}|\langle\frac{1}{2},\frac{-1}{2}||\frac{1}{2},\frac{+1}{2}\rangle|\frac{1}{2},\frac{-1}{2}\rangle = 1$$

$$\langle\frac{1}{2},\frac{+1}{2}|\langle\frac{1}{2},\frac{-1}{2}||\frac{1}{2},\frac{-1}{2}\rangle|\frac{1}{2},\frac{+1}{2}\rangle = 0$$

$$\langle\frac{1}{2},\frac{-1}{2}|\langle\frac{1}{2},\frac{+1}{2}||\frac{1}{2},\frac{+1}{2}\rangle|\frac{1}{2},\frac{-1}{2}\rangle = 0$$

Key facts

1. states with different quantum numbers (eigenvalues) are orthogonal

2. Inner product of state with itself = 1 assuming it is normalized

Putting it all together

$$|1,1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

1/2×1/2		1	1	0	1
		+1	0	0	−1
+1/2	+1/2	1			
+1/2	−1/2		1/2	1/2	
−1/2	+1/2		1/2	-1/2	
−1/2	−1/2				1

- These coefficients are called “Clebsch-Gordan” coefficients
- Some poor person has worked all the coefficients so that we don’t have to.
- You just have to know how to read the table
  - By convention all entries have an implied squared root:

Practice with Clebsch-Gordan Coefficients:

+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

+1	−1	1
0	0	3
−1	+1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

- Use the Clebsch-Gordon coefficient to decompose the states of the two spin 1 systems.
  - How many states are there?
  - What are their J values?
  - Perform the highest weight decomposition of the J=2 state and check with the table.
  - Determine the J=1, J_z=1 using orthogonality and check with the table.

The Pauli Matrices

- Define the matrices:

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- At home, you should convince yourself that:
  - these matrices satisfy the commutation relations $[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$
  - the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenvectors of S_z with the appropriate eigenvalues
  - operators S_+ and S_- have the desired properties.
- all states of this system have the appropriate eigenvalue for a spin 1/2 system for the operator S^2.

The Pauli Matrices

- Define the matrices corresponding to our S_i operators

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- and our eigenvectors corresponding to our eigenstates of S, S_z.

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dirac notation	Pauli matrix notation
$S_z\|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \frac{\hbar}{2}\|\tfrac{1}{2}, \tfrac{1}{2}\rangle$	$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$S_z\|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\frac{\hbar}{2}\|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$	$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

More on Pauli Matrices:

- Symbolic vs. Matrix form

$$S_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$$

$$S_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}\tfrac{3}{2} - \tfrac{1}{2}\tfrac{-1}{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}\tfrac{3}{2} - \tfrac{-1}{2}\tfrac{1}{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0$$

$$S_+ = S_x + iS_y = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_- = S_x - iS_y = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

Total Angular Momentum

$$S^2|\tfrac{1}{2},\tfrac{1}{2}\rangle = l(l+1)\hbar^2|\tfrac{1}{2},\tfrac{1}{2}\rangle = \tfrac{3}{4}\hbar^2|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$S^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle = l(l+1)\hbar^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{3}{4}\hbar^2|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4} \times \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \right]$$

$$\frac{3\hbar^2}{4} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{3\hbar^2}{4} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3\hbar^2}{4} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{3\hbar^2}{4} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3\hbar^2}{4} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Summary:

- Symmetries are a fundamental concept in particle physics
  - Nöther: symmetries ↔ conservation laws
  - symmetry operations in physics can often be expressed algebraically as matrices
- Angular momentum conservation arises from the isotropy/rotational symmetry
  - Non-trivial commutation relations between L_x, L_y, L_z
    - we can diagonalize only with respect to one
    - L^2 however, commutes with L_i, so simultaneous eigenstates exist
  - Raising/lowering operator allows one to fill out all the states of a given angular momentum when we add two components of angular momentum
    - emply "highest weight decomposition" with the highest L, L_z state
    - Clebsch-Gordan coefficients: relation between eigenstates of the combined system vs. eigenstates of the component systems
  - Pauli matrices: explicit representation of the 2-state spin 1/2 system