Relativistic Kinematics

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Announcements

- I will be out of town starting Wednesday afternoon
  - Thursday lecture will be given by Prof. W. Trischuk
  - I will have to cancel my office hours this week
    - please do not hesitate to email me with any questions, etc.

Overview

- Preview of what kind of problems we will deal with
  - decay
  - scatter
- Review of our tools
  - 4-momentum conservation, invariants
  - reference frames
- Notation
- Relation between kinematic quantities
- A few examples

Notation

- Note:
  - for the most part, we deal with four-momentum
    - they may or may not be labeled by their Lorentz index
    - “momentum p” implicitly means four-momentum p
  - 3-momentum will be explicitly labeled
    - either by bold font, arrows, Roman indices
  - this notation carries over to the vector algebra
    - when there is a dot product, if the quantities are not bold/arrowed, it refers to a Lorentz-invariant product
    - otherwise, it is a three-vector product.

Relations:

- Some basic kinematic relations:
  - Energy, mass, momentum: from our definition

$$p^\mu = (E/c, \vec{p}) = m\gamma(c, \vec{v}) \qquad E = \gamma mc^2 \qquad \vec{p} = \gamma m\vec{v}$$

$$E^2 - \mathbf{p}^2c^2 = \gamma^2 m^2(c^4 - \mathbf{v}^2c^2) = \frac{m^2c^4(1-\mathbf{v}^2/c^2)}{1-\mathbf{v}^2/c^2} = m^2c^4$$

$$p^2 = (E/c)^2 - \mathbf{p}^2 = m^2c^2$$

  - energy/momentum and velocity

$$\vec{p} = \gamma m\vec{v} \qquad E = \gamma mc^2$$

$$\frac{|\mathbf{p}c|}{E} = \frac{\gamma m\mathbf{v}c}{\gamma mc^2} = \frac{v}{c} = \beta$$

v

2-body decay at rest (I)

- Perhaps the easiest kinematic situation
  - e.g. $K^{*0}(1) \rightarrow K^{+}(2) + \pi^{-}(3)$, calculate outgoing energies
    - calculate the energy of particle 2
  - note that we don't really know anything except the masses
  - Approach one: by conservation of energy and momentum (separately):

$$E_1 = E_2 + E_3 = m_1 c^2 \qquad \mathbf{p_1} = \mathbf{p_2} + \mathbf{p_3}$$

$$E_2 + \sqrt{\mathbf{p_2}^2 c^2 + m_3^2 c^4} = m_1 c^2$$

$$m_1^2 c^4 + E_2^2 - 2E_2 m_1 c^2 = \mathbf{p_2}^2 c^2 + m_3^2 c^4$$

$$m_1^2 c^4 + m_2 c^4 - 2E_2 m_1 c^2 = m_3^2 c^4$$

$$E_2 = \frac{m_1^2 c^2 + m_2 c^2 - m_3^2 c^2}{2m_1}$$

2-body decay (II)

- Approach 2: with 4-momentum algebra
- From conservation:

$$p_1 = p_2 + p_3 \longrightarrow p_1^2 = (p_2 + p_3)^2 = p_2^2 + p_3^2 + 2p_2 \cdot p_3$$

- translate these into masses:

$$\frac{m_1^2c^2 - m_2^2c^2 - m_3^2c^2}{2} = p_2 \cdot p_3$$

- Reuse conservation:

$$p_2 \cdot p_3 = p_2 \cdot (p_1 - p_2) = p_1 \cdot p_2 - m_2^2c^2$$

- explicitly evaluate $p_1 \cdot p_2$ remembering that p_1 is at rest

$$p_1 \cdot p_2 = E_1E_2/c^2 - \mathbf{p_1} \cdot \mathbf{p_2} = m_1E_2$$

$$E_2 = \frac{m_1^2c^2 + m_2^2c^2 - m_3^2c^2}{2m_1}$$

Observations

- Start by assigning 4-momenta notation to the incoming and outgoing particles and setting up the 4-momentum conservation equation
- In four momentum equations:
    - squaring usually terms many of the terms into masses, which are easy to deal with (especially if they are zero)
        - it also turns it into a scalar equation
    - In the CM frame and the lab frame, there are quantities that are zero (momentum of initial particle in CM frame, etc.)
        - Use these in the four-vector expression (like a dot product) to zero out parts of the expression $$p_1 \cdot p_2 = E_1 E_2/c^2 - \mathbf{p_1} \cdot \mathbf{p_2} = m_1 E_2$$
    - Keep an eye out for opportunities to combine E^2, $\mathbf{p}^2$, m^2
        - Once you have E or **p** in you can easily translate between them (and v)

Laboratory scattering

- Consider the process A + B → C where B is at rest.
  - What energy of A required to produce C?
  - Assign labels (trivial: A → p_A, b → p_B, C → p_C)
  - Conservation of 4-momentum:

$$p_A + p_B = p_C$$

  - Square the equation:

$$(p_A + p_B)^2 = p_C^2 \qquad p_A^2 + p_B^2 + 2(p_A \cdot p_B) = p_C^2$$

$$m_A^2 c^2 + m_B^2 c^2 + 2(p_A \cdot p_B) = m_C^2 c^2$$

  - Now in the lab frame:

$$p_A = (E_A/c, \mathbf{p}_A)$$
$$p_B = (m_B c, \mathbf{0})$$

$$p_A \cdot p_B = E_A m_B$$

$$\boxed{E_A = \frac{m_C^2 c^2 - m_A^2 c^2 - m_B^2 c^2}{2 m_B}}$$

The other way:

- Consider the process A(1) + B(2) → C(3) where B is at rest.
  - What energy of A required?
- Energy/momentum conservation:

$$E_A + E_B = E_C \qquad E_A/c + m_B c = E_C/c$$

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_C \qquad \mathbf{p}_A = \mathbf{p}_C$$

$$E_A + m_B c^2 = c\sqrt{\mathbf{p}_A^2 + m_C^2 c^2}$$

- square both sides

$$E_A^2 + m_B^2 c^4 + 2E_A m_B c^2 = \mathbf{p}_A^2 c^2 + m_C^2 c^4$$

  - . . . . and so on

Application:

- What minimum energy is required for the reaction:

$$p + p \rightarrow p + p + p + \bar{p}$$

  - with one of the initial protons at rest to proceed?
  - in the lab frame, it is complicated since the outgoing products must be in motion to conserve momentum
  - in the CM frame, however, the minimum energy configuration is where the outgoing products are at rest.
    - (kinematically equivalent a single particle of mass $4m_p$)

$$E_1 = \frac{m_3^2c^2 - m_1^2c^2 - m_2^2c^2}{2m_2}$$

    - set $m_3 = 4m_p$, $m_1, m_2 = m_p$
    - $E_1 = 7\, m_pc^2$

Looking back:

- “classic” vs. “relativistic” kinematics
  - recall that in relativity, mass is a form of energy
    - it can be interconverted with other forms (kinetic, etc.) as long as momentum/energy is conserved overall. It is not separately conserved
    - in classical kinematics, mass is conserved, along with energy and momentum.
- In our simple examples, the use of four momentum algebra (perhaps) reduced the complexity of expressions, though the number of steps was ~same.
  - In more complicated examples, you will find it beneficial to use all the tools that we have (invariants, reference frames, etc.)

Compton Scattering:

- Consider the process $\gamma + e \rightarrow \gamma + e$ where the electron is initially at rest.
  - If the γ scatters by an angle θ, what is it's outgoing energy?
  - Assign labels:
    - p_1 = incoming photon, p_2 = initial electron
    - p_3 = outgoing photon, p_4 = outgoing electron
  - Conservation of 4-momentum:

$$p_1 + p_2 = p_3 + p_4 \qquad\qquad p_1 + p_2 - p_3 = p_4$$

  - Square the equation:

$$(p_1 + p_2 - p_3)^2 = p_4^2 \qquad p_1^2 + p_2^2 + p_3^2 + 2(p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) = p_4^2$$

$$m_e^2 c^2 + 2(p_1 \cdot p_2 - p_1 \cdot p_3 - p_2 \cdot p_3) = m_e^2 c^2$$

  - Now in the lab frame:

$$p_1 = (E_1/c, \mathbf{p}_1) \qquad p_1 \cdot p_2 = E_1 m_e$$

$$p_2 = (m_e c, \mathbf{0}) \qquad p_1 \cdot p_3 = E_1 E_3/c^2 - \mathbf{p}_1 \cdot \mathbf{p}_3 = E_1 E_3 (1 - \cos\theta)/c^2$$

$$p_3 = (E_3/c, \mathbf{p}_3) \qquad p_2 \cdot p_3 = E_3 m_e$$