# Lecture 3: Special Relativity

H. A. Tanaka

# Announcements

- Office hours:
  - Wednesday 1600-1700, Thursdays 1500-1600 (MP801A)
- Vince:
  - MP815
  - Wednesday 1500-1600

# Overview

- Review central postulates of special relativity
- Review their consequences
- Introduce four vectors and index notation
- Develop Lorentz algebra in terms of index notation
- Define invariant quantities
- Examine the consequences for energy/momentum in special relativity

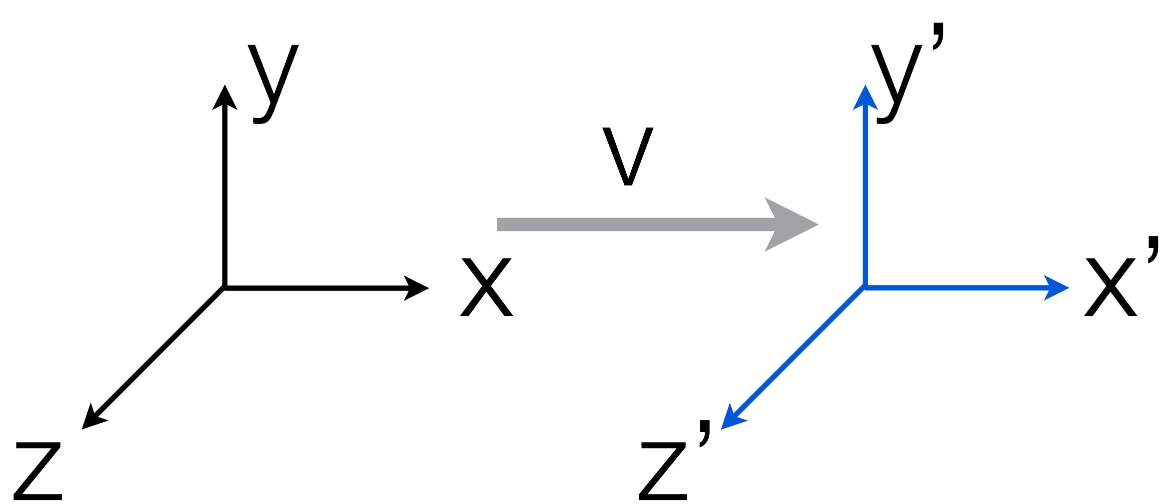
# Special Relativity

- Postulates:
  - the laws of physics are identical in all inertial reference frames.
  - the velocity of light is the same in all inertial frames
- Consequences:
  - The same speed of light will be observed regardless of whether you are moving towards it or away from it (Michelson-Morley experiment)
    - strange velocity addition properties
  - Simultaneity is relative; different in different reference frames.
  - Lorentz (length) contraction
  - Time dilation

# Lorentz Transformation

- In 3D space, we know how coordinates "transform"
- There are corresponding transformations in SR "Lorentz Transformation"
  - coordinates and time observed w.r.t. a frame moving with constant velocity w.r.t to the original frame

$$\begin{aligned} t' &= \gamma(t - \tfrac{v}{c^2}x) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \qquad \begin{aligned} t &= \gamma(t' + \tfrac{v}{c^2}x') \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}$$




$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

# Consequences

- Simultaneity:

$$t'_A - t'_B = \gamma(t_A - t_B + \frac{v}{c^2}(x_B - x_A))$$

- length of an object viewed from a moving reference frame

$$x'_A - x'_B = \gamma(x_A - x_B + v(t_B - t_A))$$

  - length in non-' system ($t_B$=$t_A$) is shorter by a factor of γ

- elapsed time viewed from a moving reference frame

$$t'_A - t'_B = \gamma(t_A - t_B + \frac{v}{c^2}(x_B - x_A))$$

  - elapsed time is shorter by factor of γ (time runs more slowly)
- etc.

# Four vectors:

- In parallel to vectors/rotations in 3D define 4-vectors/Lorentz transformation in 3D+time:
  - 3-vectors are objects that correspond to the x, y, z components of something.
  - They have definite properties under the transformation of these coordinates. What are they?
- Four vector $x^\mu$: time + spacial coordinates
  - to give them the same units, define $x^0=ct$
    - $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3=z,\ \beta=v/c$

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned} \qquad\qquad \begin{aligned} x^0 &= \gamma(x'^0 + \beta x'^1) \\ x^1 &= \gamma(x'^1 + \beta x'^0) \\ x^2 &= x'^2 \\ x^3 &= x'^3 \end{aligned}$$

symmetric form between time ($x^0$) and boosted coordinate ($x^1$ in this case)

# Matrix Form

- We can write the transformations in matrix form:

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned} \qquad \begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

  - boost along y axis? rotation?

- In practice, we usually do not write out matrices.
  - use “indices”, “summation” to express the matrix algebra

$$x^{\mu\prime} = \sum_{\nu} \Lambda^{\mu}_{\nu} x^{\nu}$$

$$x^{\mu\prime} = \Lambda^{\mu}_{0} x^0 + \Lambda^{\mu}_{1} x^1 + \Lambda^{\mu}_{2} x^2 + \Lambda^{\mu}_{3} x^3$$

  - etc.
  - we’ll see the point of the “upstairs” and “downstairs” index.

# Summation Notation

- It's also a pain to write the "sum" Σ all the time
- Einstein invented the "summation" convention:
  - if two indices are repeated with the same letter, then summation is implied

$$x^{\mu\prime} = \sum_{\nu} \Lambda^{\mu}_{\nu} x^{\nu} \rightarrow \Lambda^{\mu}_{\nu} x^{\nu}$$

  - we will apply this convention generally, not just with Lorentz Indices
  - repeated indices = "contracted", non-repeated "free"
- Also, define:
  - "contravariant" four-vector: $x^0=ct,\ x^1 = x,\ x^2 = y,\ x^3=z$
  - "covariant" four-vector: $x_0=ct,\ x_1 = -x,\ x_2 = -y,\ x_3 = -z$
- Likewise "contravariant" and "covariant" indices.
  - we'll just call them "upstairs" and "downstairs"
- The index notation also is insensitive to ordering of terms

# Metric Tensor

- Contra/covariant vectors are related by the metric tensor g

$$x^\mu = g^{\mu\nu} x_\nu \qquad\qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- the metric tensor reverses the sign of spatial components
  - it raises a downstairs index to an upstairs index
  - what about $g^\mu{}_\nu$, $g_\mu{}^\nu$, $g_{\mu\nu}$? What are these quantities?
  - Define the "invariant" quantity

$$g^{\mu\nu} x_\mu x_\nu \rightarrow x_\mu x^\mu$$

what is this quantity explicitly?

  - "invariant" because it is the same in all reference frames
    - if we Lorentz transform $x$, all the components of $x$ may change, but this combination does not change.
    - 3D analog?

# Invariants

- We can classify the invariant quantity into three categories:
  - $x^2 = x_\mu x^\mu = c^2 t^2 - \mathbf{x}^2 > 0$ (timelike)
  - $x^2 = 0$ (lightlike)
  - $x^2 < 0$ (spacelike)
- Since $x^2$ is the same in all reference frames, all observers agree on the categories
- Implications for causality:
  - spacelike events cannot be causally connected (one cannot cause the other)
  - why?

# Benefits

- What's the point of "up/downstairs", "contra/covariant"?
  - we have a funny concept of the "length" or "magnitude" of a four-vector, i.e. invariant quantities
  - relative sign between the time and space components
- The component convention allows a way of creating invariant quantities and also check equivalence of the quantities
  - think about row vectors, column vectors, matrices, etc.
  - It boils down to:
    - Expressions must have the matching free upstairs and downstairs indices ("free" = "unsummed")
    - Summed indices must be pairs of upstairs and downstairs

# Generalizing

- Take any two four-vectors (say $a$, $b$) and their product

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} \ldots$$

  - this will also be invariant with respect Lorentz transforms
    - (3D analog?)
- The indices give us a way to classify quantities and how they transform
  - invariant/"scalar": no free indices: no transformation at all
  - vector: one $\Lambda$ for the single free index

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$$

  - Tensor: one $\Lambda$ for each free index

$$\Lambda_1{}^{\mu\nu\prime} \rightarrow \Lambda_2{}^\mu_\rho \Lambda_2{}^\nu_\sigma \Lambda_1{}^{\rho\sigma} \qquad x^{\mu\prime} y^{\nu\prime} = \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho y^\sigma$$

  - take any of these expressions and move indices up/down
    - (put in a g with a repeated index if it helps make things clearer)

# Momentum:

- We have dealt with one kind of four vector (coordinate)
  - Are there others?
- We can construct the “energy/momentum” four vector by considering the proper time:

$$\tau = \frac{t}{\gamma}$$

  - this corresponds the elapsed time in the rest frame of whatever system you are looking at.
  - If we take derivatives of the space-time coordinates wrt $\tau$, we end up with the quantities

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma(\frac{d(ct)}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = \gamma(c, \vec{v})$$

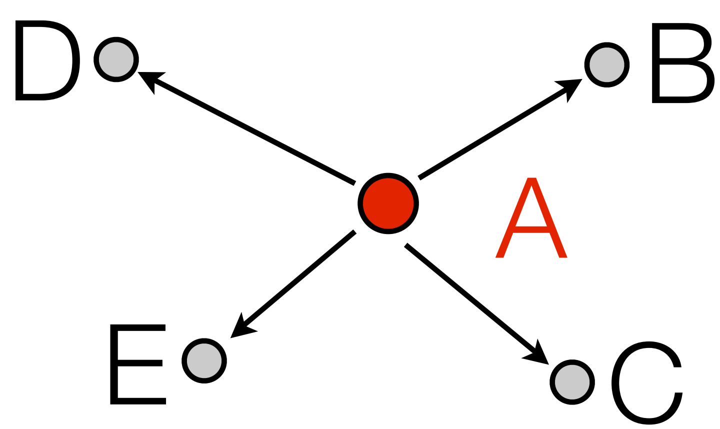
  - it’s easy to show that $\eta^\mu \eta_\mu$ = c$^2$ $\rightarrow$ $\eta^\mu$ is a four-vector

# Energy/Momentum

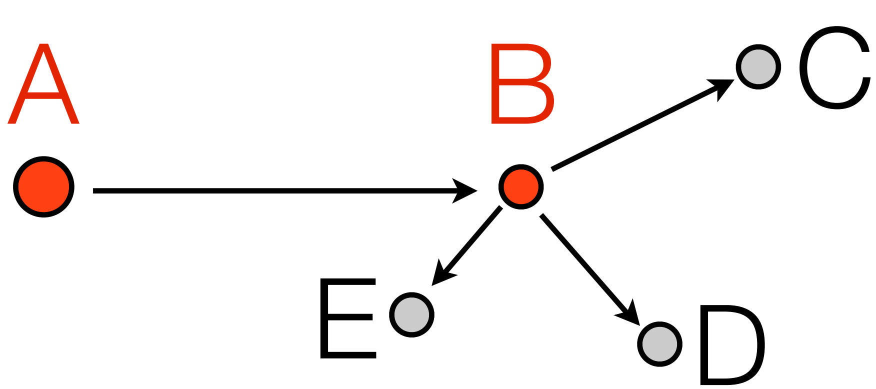
- Now multiply $\eta^\mu$ by the mass of the object to define $p^\mu$
  - $p^\mu = (\gamma mc, \gamma m\boldsymbol{v}) = (E/c, \mathbf{p})$
  - this defines the energy and momentum of an object (4-momentum) with invariant product $p^\mu p_\mu = m^2c^2$
  - we find that these quantities are conserved
    - How do we know this?
    - each component of the 4-momentum is the same before and after a process or reaction
- We will find that there are other four vectors
  - electric charge and current density
  - particle number and flow density

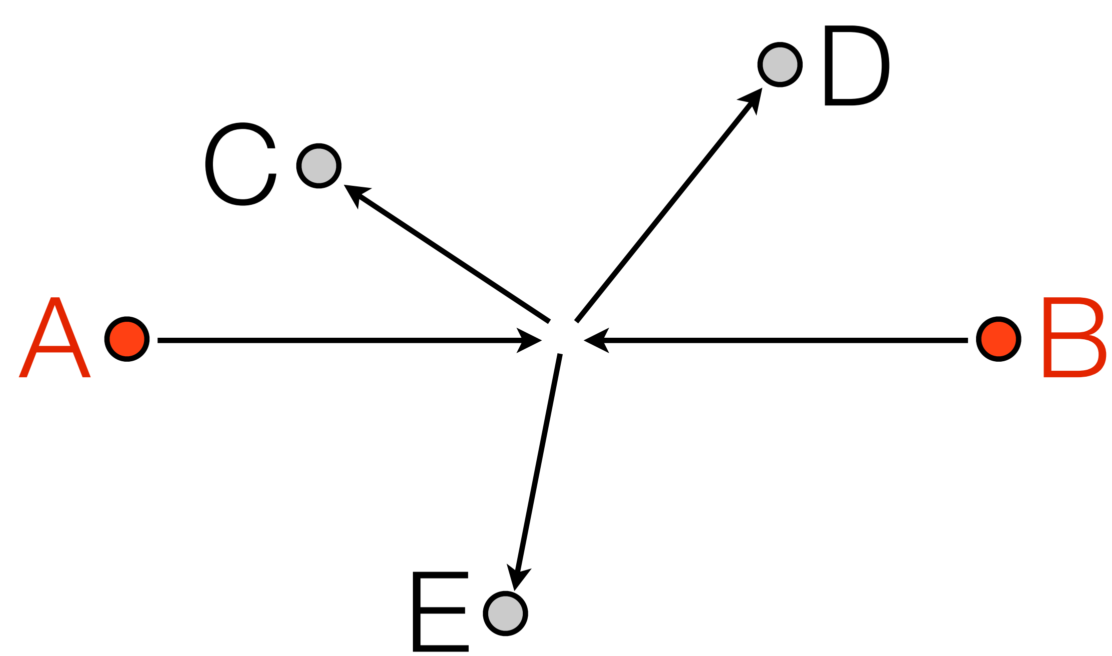
# Decays and Scatters

- We will typically consider two kinds of processes:
- Decays: A→B+C+ . .




- Scattering: A + B → C + D + E +  . .




- “What is the energy/momentum of the outgoing particles?”

# Basic Tools: Conservation

why i,j different but μ the same?

- Energy conservation:

$$\sum_i E_I^i = \sum_j E_F^j$$

  - $I$ = “initial”, $F$ =“final”

$$\sum_i p^{i\,\mu}_{\ I} = \sum_j p^{j\,\mu}_{\ F}$$

$$\sum_i p^{i}{}_{I} = \sum_j p^{j}{}_{F}$$

- Momentum conservation:

$$\sum_i p^i_{x\,I} = \sum_j p^j_{x\,F}$$

$$\sum_i p^i_{y\,I} = \sum_j p^j_{y\,F}$$

$$\sum_i p^i_{z\,I} = \sum_j p^j_{z\,F}$$

$$\sum_i \vec{p}^{\,i}_I = \sum_j \vec{p}^{\,j}_F$$

Four equations relating the initial and final state energies and momenta

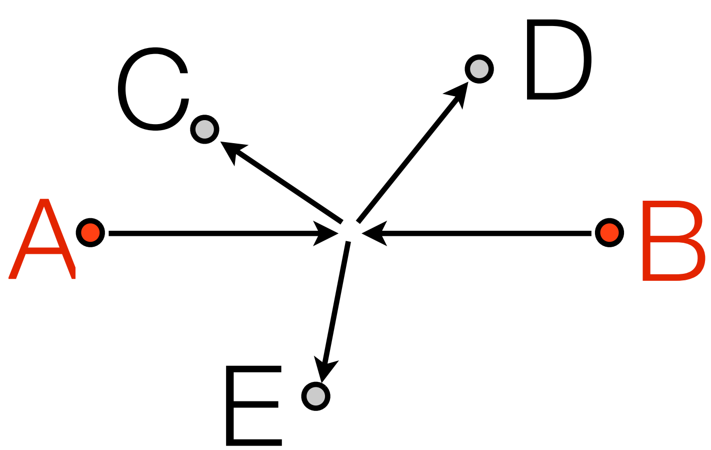
# Invariance:

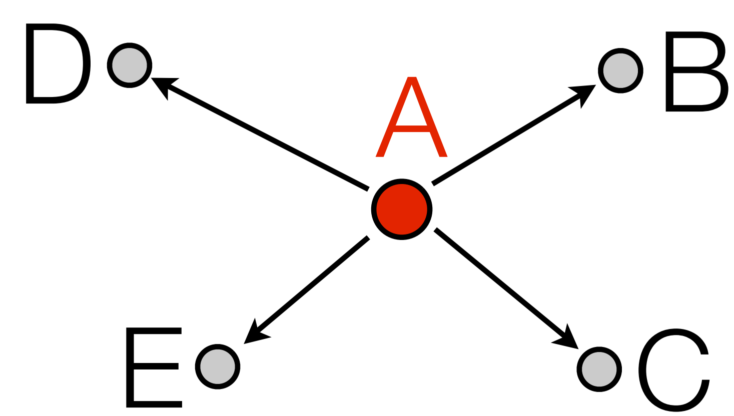
- We discussed the "dot product" of two four vectors:
    - $a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$
    - $= a^\mu b_\mu$, $a_\mu b^\mu$, $g_{\mu\nu} a^\mu b^\nu$, etc.
- Explicitly in terms of two 4-momentum vectors:
    - $p_1 \cdot p_2 = p_1{}^0 p_2{}^0 - p_1{}^1 p_2{}^1 - p_1{}^2 p_2{}^2 - p_1{}^3 p_2{}^3 = E_1 E_2 / c^2 - \mathbf{p_1} \cdot \mathbf{p_2}$
    - the dot product of a four momentum with itself:
    - $p_1 \cdot p_1 = p_1{}^2 = (E_1/c)^2 - \mathbf{p_1}^2 =$ . . . . .
- Invariants are useful because
    - they are the same in all reference frames
    - reduces multicomponent equation to scalar quanties
    - they may save you from having to explicitly evaluate by choosing a reference frame/coordinate frame
    - If massless particles are involved, it will eliminate terms

# Reference Frames

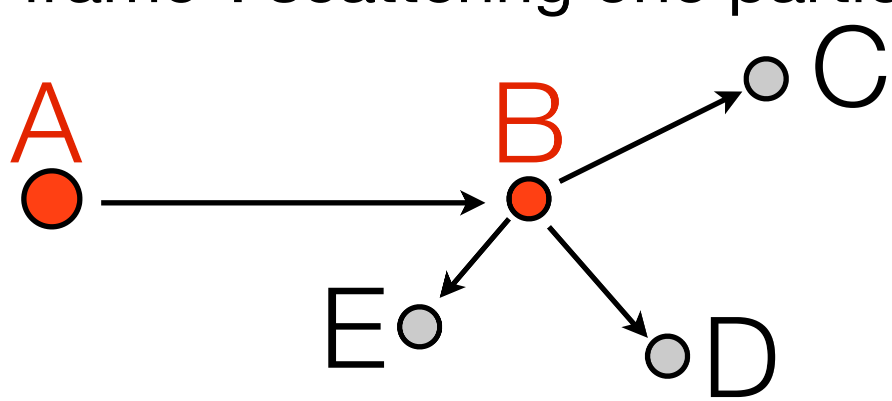
- We will typically operate in two kinds of reference frames
- "Center-of-momentum":
  - sum of momentum is zero
  - e.g. colliding beams, decay at rest

$$\sum_i \vec{p}_I^{\,i} = 0$$




- "Lab frame": scattering one particle at rest:



# Notation

- Note:
  - for the most part, we deal with four-momentum
    - they may or may not be labeled by their Lorentz index
    - “momentum p” implicitly means four-momentum p
  - 3-momentum will be explicitly labeled
    - either by bold font or arrows
  - this notation carries over to the vector algebra
    - when there is a dot product, if the quantities are not bold/arrowed, it refers to a Lorentz-invariant product
    - otherwise, it is a three-vector product.

# Summary

- I hope the basic concepts of SR are already familiar to you
  - Postulates, consequences
  - Lorentz transformations
  - invariant quantities
- What may be new to you is the index notation and the associated algebra
  - Organize all the algebra and ensuring that all the things come out correctly
  - Tells us what kind of quantity we are dealing with, what we can do with it.
  - Makes the process “mechanical” so we don’t have to worry about it.
- In addition to space-time 4-vectors, we have energy-momentum 4-vectors
  - 0th component is energy, 1-3 components are (3) momentum
  - invariant quantity is the mass-squared of the particle
  - we will primarily deal with energy-momentum 4-vectors in the class to express kinematic constraints (energy, momentum conservation, etc.).