# $e^+ + e^- \longrightarrow \tau^+ + \tau^-$

H. A. Tanaka

# Midterm

- Next Thursday in class
- Covers Chapters 1-6
    - Basic interaction properties
        - drawing Feynman diagrams for a process
        - which processes are allowed, favoured, etc. (interaction, phase space, CKM matrix element, etc.)
    - Special relativity, relativistic kinematics
    - Isospin, Parity, CP violation
    - Basic phase space.
- A formula sheet will be provided with relevant information
- You can also bring a scientific calculator.

# Other

- Final examination:
  - Monday, 21 December 1400-1700.
  - SS 2118 (Sidney Smith Hall, 100 St. George Street)
- Problem Set 3 due today at 1700 in drop box.

# $e^+ + e^- \to \tau^+ + \tau^-$

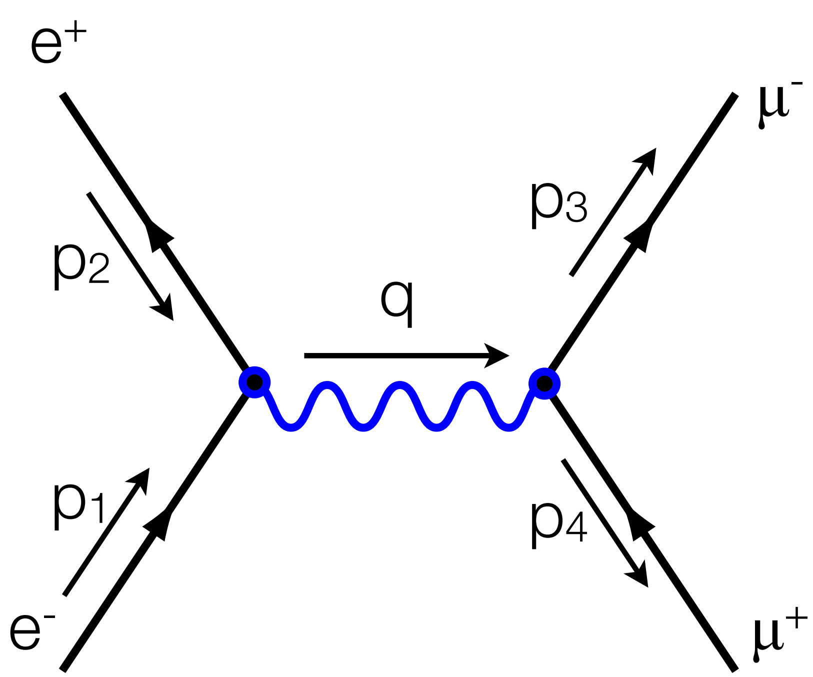
- Calculate the spin averaged cross section for this process in the CM frame as a function of the incoming electron/positron energy.
- Let's call the electron mass m, $\tau$ mass M
  - i.e. don't assume the particles are massless
  - $\tau$ is a spin 1/2 fermion just like an electron; Feynman rules are the same as an electron, just with a different mass

$$\langle |\mathcal{M}|^2 \rangle = g_e^4 \left\{ 1 + \left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)\cos^2\theta + \frac{m^2c^4}{E^2} + \frac{M^2c^4}{E^2} \right\}$$

# DIO

- Consider the process $e^+e^- \to l^+l^-$ where $l$ is a muon or tau
  - Assume energies are high enough that $m_e/m_\mu \sim 0$.
- Step I: Write down the Feynman diagram(s) for this process, labeling the momenta of the particles (incoming, outgoing and virtual)
- Step II: Use the Feynman rules to write down an expression for the amplitude.
- Step III: Sum over the spins of both the initial-state and final-state particles to obtain a expression for $|M|^2$ in terms of the traces of $\gamma$ matrices.
  - Note $\bar{\gamma^\mu} = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$
- Step IV: Use the trace relations to obtain $|M|^2$ in terms of the dot products of the four-momenta and the masses of the particles
- Step V: Assume that you are in the CM frame with the incoming $e^+e^-$ coming along the z axis. Express $|M|^2$ in terms of the energy of the $e^+e^-$ , the masses of the particles, and and the angle of the outgoing $l^-$ relative to the $e^-$.

# Step I/II: The Feynman Diagram and rules




$$\frac{1}{(2\pi)^4}\int d^4q\ \frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}(3)\ ig_e\gamma^\mu\ v(4) \quad (2\pi)^4\delta^4(q-p_3-p_4)$$

$$\bar{v}(2)\ ig_e\gamma^\nu\ u(1) \quad (2\pi)^4\delta^4(p_1+p_2-q)$$

$$[\bar{u}(3)\ \gamma^\mu\ v(4)]\ g_{\mu\nu}\ [\bar{v}(2)\ \gamma^\nu\ u(1)]$$

$$i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\frac{g_e^2}{(p_1+p_2)^2}$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}[\bar{u}(3)\ \gamma^\mu\ v(4)]\,[\bar{v}(2)\ \gamma_\mu\ u(1)]$$

# Step III: Summing over spins:

- To get $|M|^2$ we need to take the complex conjugate of the M:

$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\left[\bar{u}(3)\,\gamma^\mu\; v(4)\right]\left[\bar{v}(2)\,\gamma_\mu\; u(1)\right]$$

$$\mathcal{M}^* = -\frac{g_e^2}{(p_1+p_2)^2}\left[\bar{u}(3)\gamma^\nu v(4)\right]^*\left[\bar{v}(2)\gamma_\nu u(1)\right]^*$$

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4}\left[\bar{u}(3)\gamma^\mu v(4)\right]\left[\bar{u}(3)\gamma^\nu v(4)\right]^*\left[\bar{v}(2)\gamma_\mu u(1)\right]\left[\bar{v}(2)\gamma_\nu u(1)\right]^*$$

$$\sum_{\text{spins}}\left[\bar{u}(3)\gamma^\mu v(4)\right]\left[\bar{u}(3)\gamma^\nu v(4)\right]^* = \mathrm{Tr}\left[(\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)\right]$$

$$\sum_{\text{spins}}\left[\bar{v}(2)\gamma^\mu u(1)\right]\left[\bar{v}(2)\gamma^\nu u(1)\right]^* = \mathrm{Tr}\left[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)\right]$$

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4}\mathrm{Tr}\left[\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)\right]\mathrm{Tr}\left[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)\right]$$

# Step IV:

- Expand the trace expressions

$$\mathrm{Tr}\left[\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)\right] = \mathrm{Tr}\left[\gamma^\mu \not{p}_4\gamma^\nu \not{p}_3 - M^2c^2\gamma^\mu\gamma^\nu\right]$$

$$\mathrm{Tr}\left[\gamma_\mu(\not{p}_1 - mc)\gamma_\nu(\not{p}_2 + mc)\right] = \mathrm{Tr}\left[\gamma_\mu \not{p}_1\gamma_\nu \not{p}_2 - m^2c^2\gamma^\mu\gamma^\nu\right]$$

- An apply the trace relations

$$\mathrm{Tr}\left[\gamma^\mu \not{p}_4\gamma^\nu \not{p}_3 - M^2c^2\gamma^\mu\gamma^\nu\right] = 4\times\left[p_4^\mu p_3^\nu + p_3^\mu p_4^\nu - g^{\mu\nu}(p_4\cdot p_3) - M^2c^2g^{\mu\nu}\right]$$

$$\mathrm{Tr}\left[\gamma_\mu \not{p}_1\gamma_\nu \not{p}_2 - m^2c^2\gamma^\mu\gamma^\nu\right] = 4\times\left[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - (p_1\cdot p_2)g_{\mu\nu} - m^2c^2g_{\mu\nu}\right]$$

- Carry out the contraction between the Lorentz indices:

$$16\times$$

$$\begin{aligned}
&(p_4\cdot p_1)(p_3\cdot p_2) + (p_4\cdot p_2)(p_3\cdot p_1) - (p_1\cdot p_2)(p_4\cdot p_3) - m^2c^2(p_4\cdot p_3)\\
&(p_3\cdot p_1)(p_4\cdot p_2) + (p_3\cdot p_2)(p_4\cdot p_1) - (p_1\cdot p_2)(p_4\cdot p_3) - m^2c^2(p_4\cdot p_3)\\
-&(p_1\cdot p_2)(p_4\cdot p_3) - (p_2\cdot p_1)(p_4\cdot p_3) + 4(p_4\cdot p_3)(p_1\cdot p_2) + 4m^2c^2(p_3\cdot p_4)\\
&-M^2c^2\left[(p_1\cdot p_2) + (p_2\cdot p_1) - 4(p_1\cdot p_2) - 4m^2c^2\right]
\end{aligned}$$

$$\begin{aligned}
16\times[&2(p_1\cdot p_4)(p_2\cdot p_3) + 2(p_1\cdot p_3)(p_2\cdot p_4) + 2m^2c^2(p_3\cdot p_4)\\
&+2M^2c^2(p_1\cdot p_2) + 4m^2c^2M^2c^2]
\end{aligned}$$

# Step IV (continued)

- Put it all together:

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4} \mathrm{Tr}\left[\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)\right] \mathrm{Tr}\left[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)\right]$$

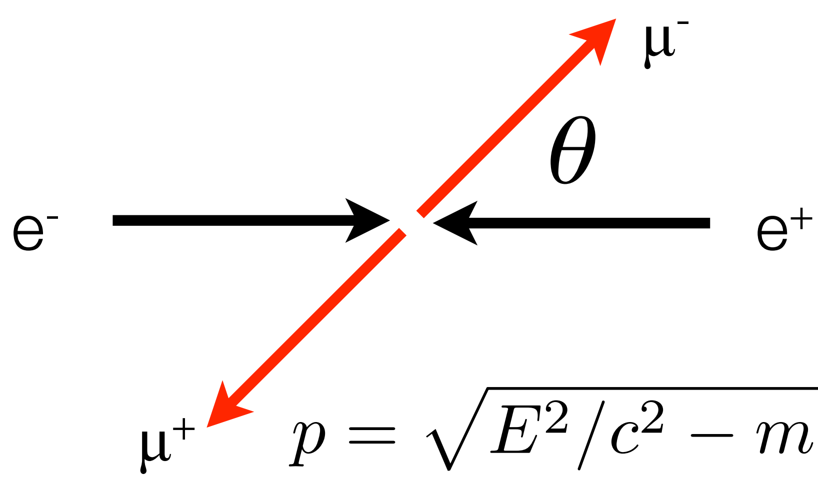
$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4} 32 \times [(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m^2c^2(p_3 \cdot p_4) + M^2c^2(p_1 \cdot p_2) + 2m^2c^2M^2c^2]$$

- Since we are averaging over the initial spins, we need to divide by 4:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(p_1+p_2)^4} 8 \times [(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m^2c^2(p_3 \cdot p_4) + M^2c^2(p_1 \cdot p_2) + 2m^2c^2M^2c^2]$$

# Step V: The Kinematics:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(p_1+p_2)^4} 8 \times [(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) + m^2c^2(p_3\cdot p_4) + M^2c^2(p_1\cdot p_2) + 2m^2c^2M^2c^2]$$




$$p = \sqrt{E^2/c^2 - m^2c^2}$$

$$p = \sqrt{E^2/c^2 - m^2c^4}$$

$$p' = \sqrt{E^2/c^2 - M^2c^2}$$

$$p_1 = (E/c, 0, p)$$

$$p_2 = (E/c, 0, -p)$$

$$p_3 = (E/c, p'\sin\theta, p'\cos\theta)$$

$$p_4 = (E/c, -p'\sin\theta, -p'\cos\theta)$$

$$\langle|\mathcal{M}|^2\rangle = g_e^4\left[1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2 + \left[1-\left(\frac{mc^2}{E}\right)^2\right]\left[1-\left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right]$$

# Working out each term:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(p_1+p_2)^4} 8 \times [(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) +$$
$$m^2c^2(p_3 \cdot p_4) + M^2c^2(p_1 \cdot p_2) + 2m^2c^2M^2c^2]$$

$$(p_1+p_2)^2 = (m^2c^2 + m^2c^2 + 2p_1 \cdot p_2) = 2(m^2c^2 + E^2c/^2 + p^2) = 4(E^2/c^2)$$

$$(p_1 \cdot p_2) = E^2/c^2 + p^2$$
$$(p_1 \cdot p_3) = E^2/c^2 - pp' \cos\theta$$
$$(p_1 \cdot p_4) = E^2/c^2 + pp' \cos\theta$$
$$(p_2 \cdot p_3) = E^2/c^2 + pp' \cos\theta$$
$$(p_2 \cdot p_4) = E^2/c^2 - pp' \cos\theta$$
$$(p_3 \cdot p_4) = E^2/c^2 + p'^2$$

$$(p_1 \cdot p_4)(p_2 \cdot p_3) = (E^2/c^2 + pp' \cos\theta)^2$$

$$(p_1 \cdot p_3)(p_2 \cdot p_4) = (E^2/c^2 - pp' \cos\theta)^2$$

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(4E^2/c^2)^2} 8 \times [2(E/c)^4 + 2p^2p'^2\cos^2\theta + m^2c^2(E^2/c^2 + p'^2)$$
$$+M^2c^2(E^2/c^2 + p^2) + 2m^2c^2M^2c^2]$$

# Endgame:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(4E^2/c^2)^2} 8 \times [2(E/c)^4 + 2p^2p'^2\cos^2\theta + m^2c^2(E^2/c^2 + p'^2)$$
$$+M^2c^2(E^2/c^2 + p^2) + 2m^2c^2M^2c^2]$$

$$p^2 = E^2/c^2 - m^2c^2$$

$$p'^2 = E^2/c^2 - M^2c^2$$

$$p^2p'^2 = \left(\frac{E}{c}\right)^4 \left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)$$

$$m^2c^2(E^2/c^2 + p'^2) + m^2c^2M^2c^2 = 2m^2E^2$$

$$M^2c^2(E^2/c^2 + p^2) + m^2c^2M^2c^2 = 2M^2E^2$$

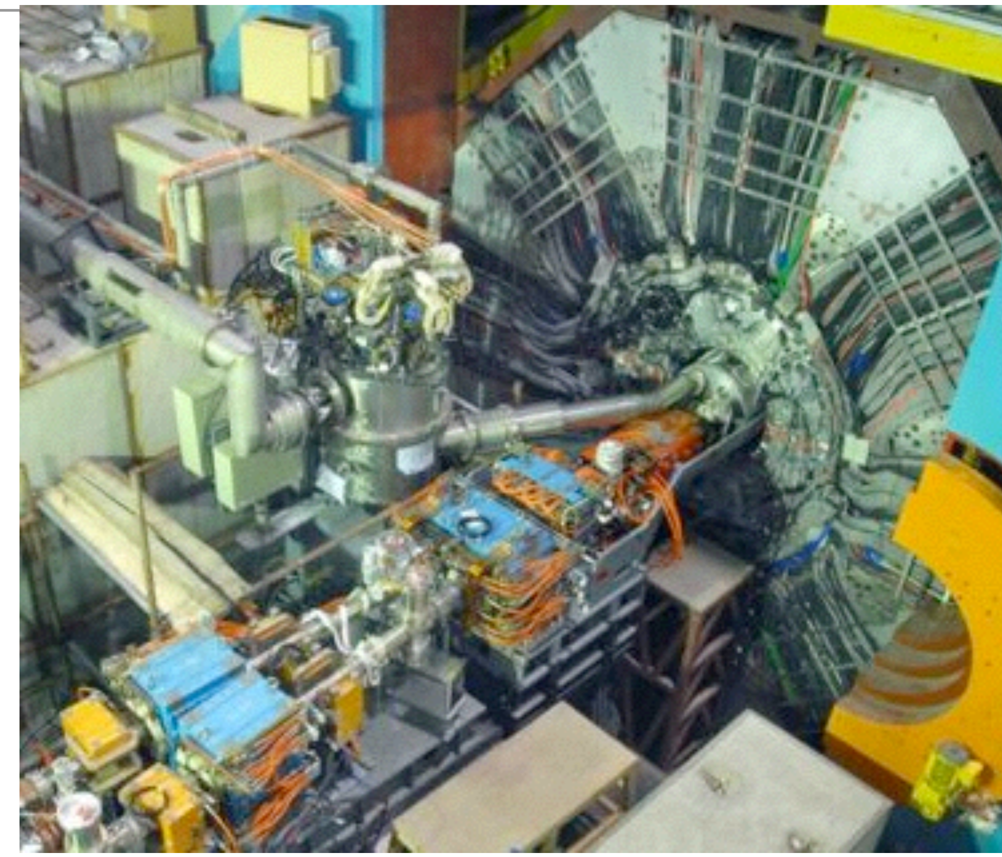
$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(4E^2/c^2)^2} 8 \times \left\{2\left(\frac{E}{c}\right)^4 \left[1 + \left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)\cos^2\theta\right] + 2m^2E^2 + 2M^2E^2\right\}$$

$$\langle|\mathcal{M}|^2\rangle = g_e^4 \left\{1 + \left(1 - \frac{m^2c^4}{E^2}\right)\left(1 - \frac{M^2c^4}{E^2}\right)\cos^2\theta + \frac{m^2c^4}{E^2} + \frac{M^2c^4}{E^2}\right\}$$

# Electron/Positron Machines around the World

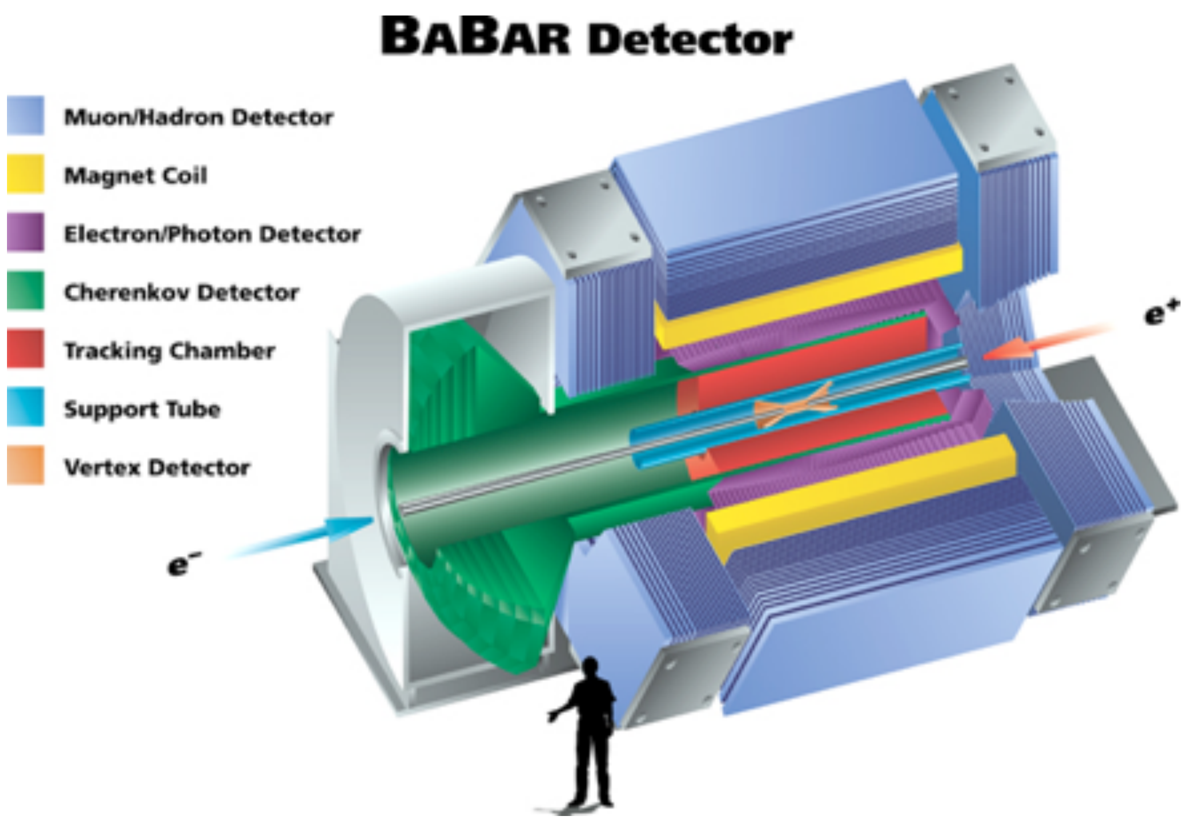

- In the US:
  - SLAC (SPEAR, PEP, PEP-II)
  - CESR
- Older machines “retire” to become synchrotron radiation sources

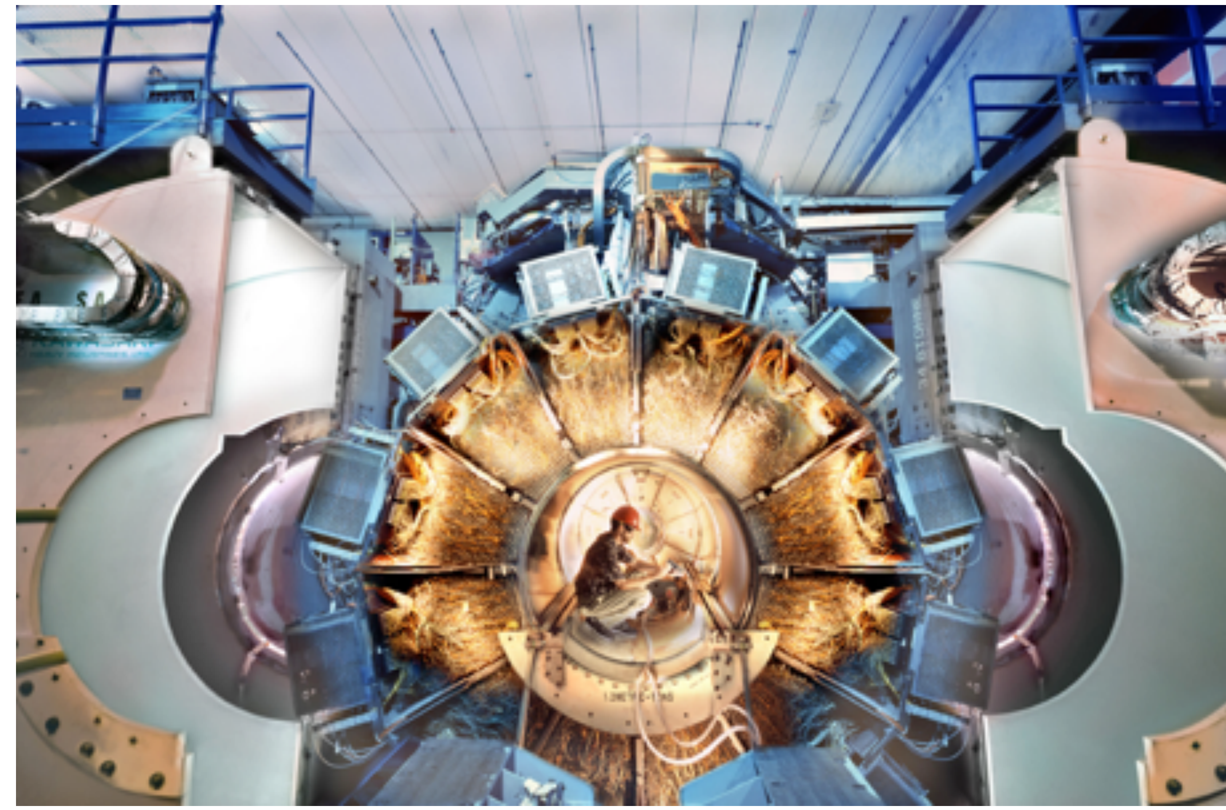
# Elsewhere:



- Left: KEK-B ring at KEK (Tsukuba, Japan)
- Top: BES spectrometer (Beijing, China)
- Other machines:
  - PETRA at DESY (Hamburg, Germany)

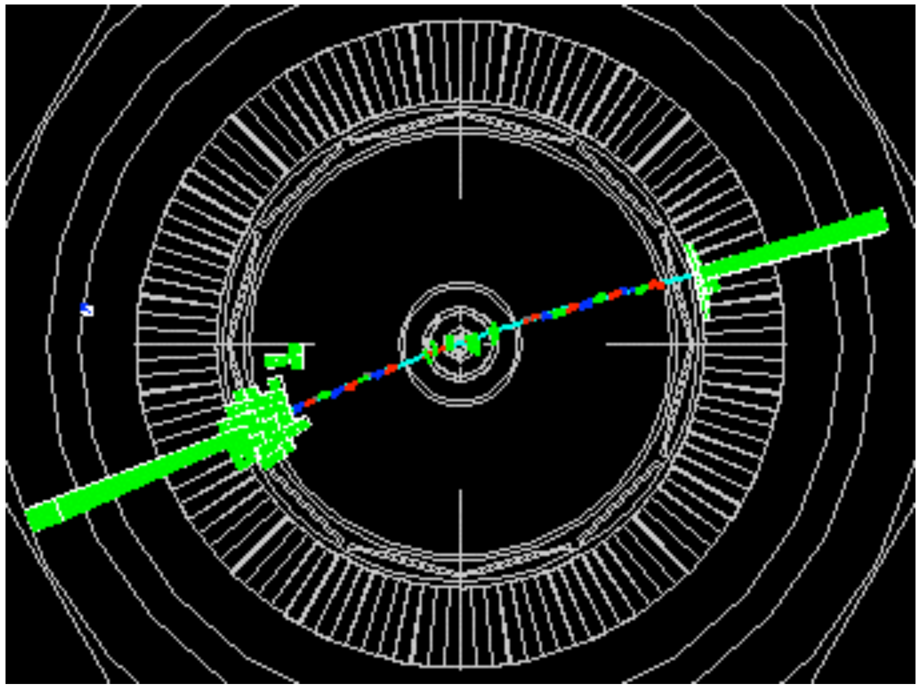
# Detectors



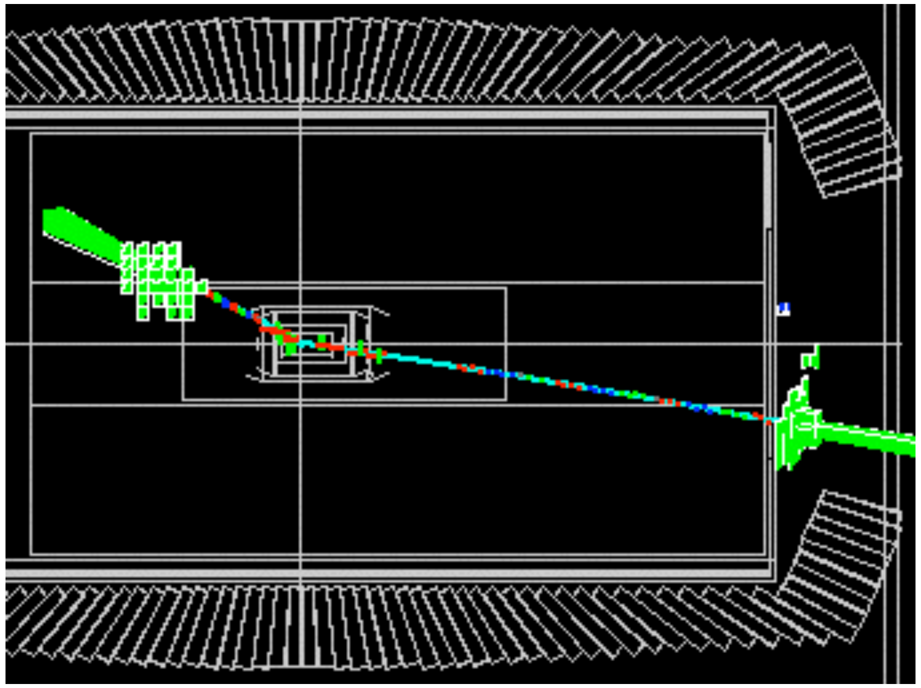



- Most detectors share a similar "cylindrical onion" design
  - Inner tracking region (silicon, drift chambers)
  - Electromagnetic calorimetry (measure and identify electron/photon energy)
  - Muon detector: identify muons by their penetration through lots of material

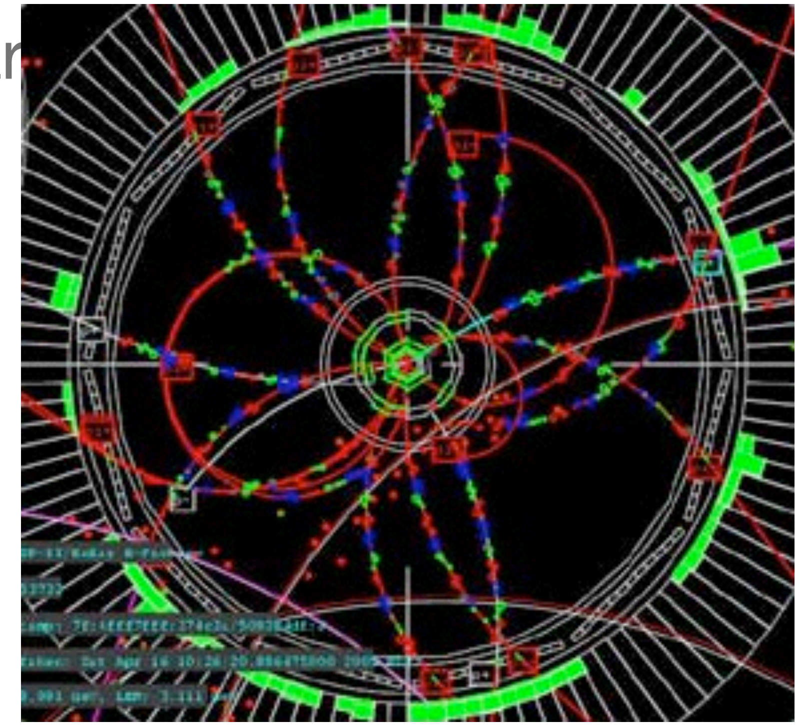
# Events at BaBar



- $e^{+}+e^{-} \rightarrow e^{+}+e^{-}$ event at BaBar (Bhabha scattering)
- Note “straightness” of tracks:
- Large deposition in electromagnetic calorimeter
- $e^{+}+e^{-} \rightarrow \mu^{+}+\mu^{-}$ would look similar, but without large energy deposition in the calorimeter



- “Hadronic” event at BaBar
- Particles like b, c quarks produced which initiate a decay chain
- “Full reconstruction” sometimes possible



# Now some physics:

- Wed derived the amplitude for $e^+e^- \to l^+l^-$

$$\langle|\mathcal{M}|^2\rangle = g_e^4 \left[1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2 + \left[1 - \left(\frac{mc^2}{E}\right)^2\right]\left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right]$$

- m = electron mass, M = lepton mass. Let's ignore the electron mass (E large enough that ($mc^2$/E) is very small:

$$\langle|\mathcal{M}|^2\rangle = g_e^4 \left[1 + \left(\frac{Mc^2}{E}\right)^2 + \left[1 - \left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right]$$

- Recalling our cross section formula:

$$\frac{d\sigma}{d\cos\theta d\phi} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\langle|\mathcal{M}|^2\rangle}{4E^2}\frac{|p_f|}{|p_i|}$$

- Integrate over the $\theta$, $\phi$ to obtain the total cross section:

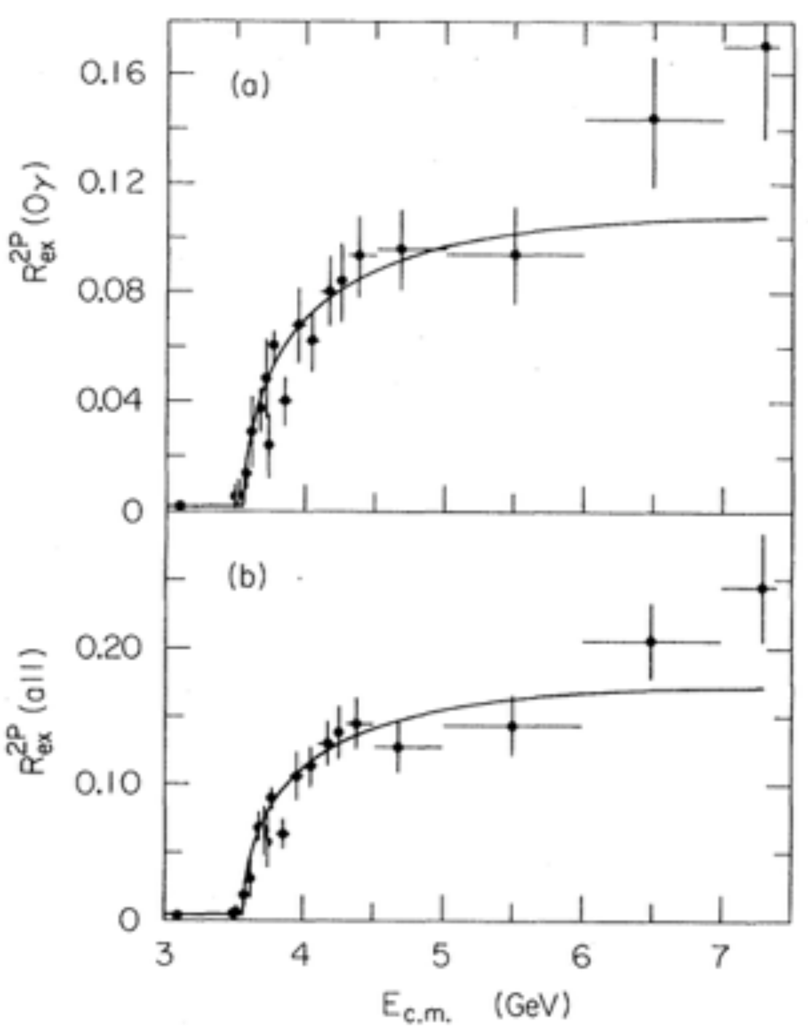
$$\sigma = \frac{\pi}{3}\left(\frac{\hbar c\alpha}{E}\right)^2 \sqrt{1-(Mc^2/E)^2}\left[1 + \frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right]$$

# Ratio of cross sections:

- $e^+ + e^- \rightarrow \mu^+ + \mu^-$ has a very distinct signature in the detector
- "Normalize" $e^+ + e^- \rightarrow \tau^+ + \tau^-$ in the detector by taking the ratio:

$$R_{\tau\mu} = \frac{\sigma_{\tau^+\tau^-}}{\sigma_{\mu^+\mu^-}} = \frac{\sqrt{1-(M_\tau c^2/E)^2}}{\sqrt{1-(M_\mu c^2/E)^2}} \times \frac{1+\frac{1}{2}(M_\tau c^2/E)^2}{1+\frac{1}{2}(M_\mu c^2/E)^2}$$

- Note: numerator is imaginary when $E < M_\tau c^2$: this is a threshold requirement




step E, count $\tau^+ + \tau^-$ and $\mu^+ + \mu^-$ events

- Ratio is effectively $R_{\tau\mu}$

- Energy $R_{\tau\mu}(E)$ depends on the spin of the $\tau$:
  - If the particle were a scalar or vector, it would have a different E-dependence
- Measures $\tau$ mass:

W. Bacino et al.

# Angular Distribution

- From our amplitude expression:

$$\langle|\mathcal{M}|^2\rangle = g_e^4\left[1+\left(\frac{Mc^2}{E}\right)^2+\left[1-\left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right]$$

- if we go to even higher energies E >> Mc$^2$, we obtain the simple form:

$$\langle|\mathcal{M}|^2\rangle = g_e^4\left[1+\cos^2\theta\right]$$

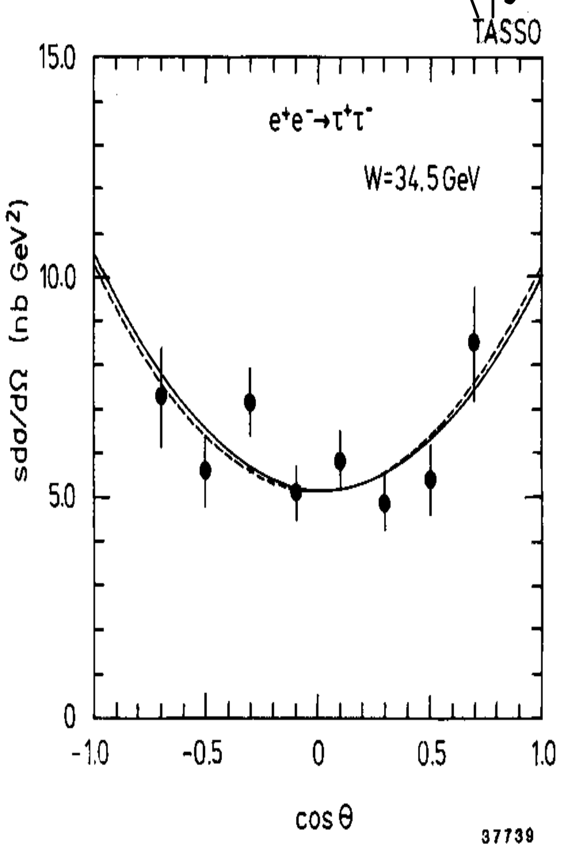


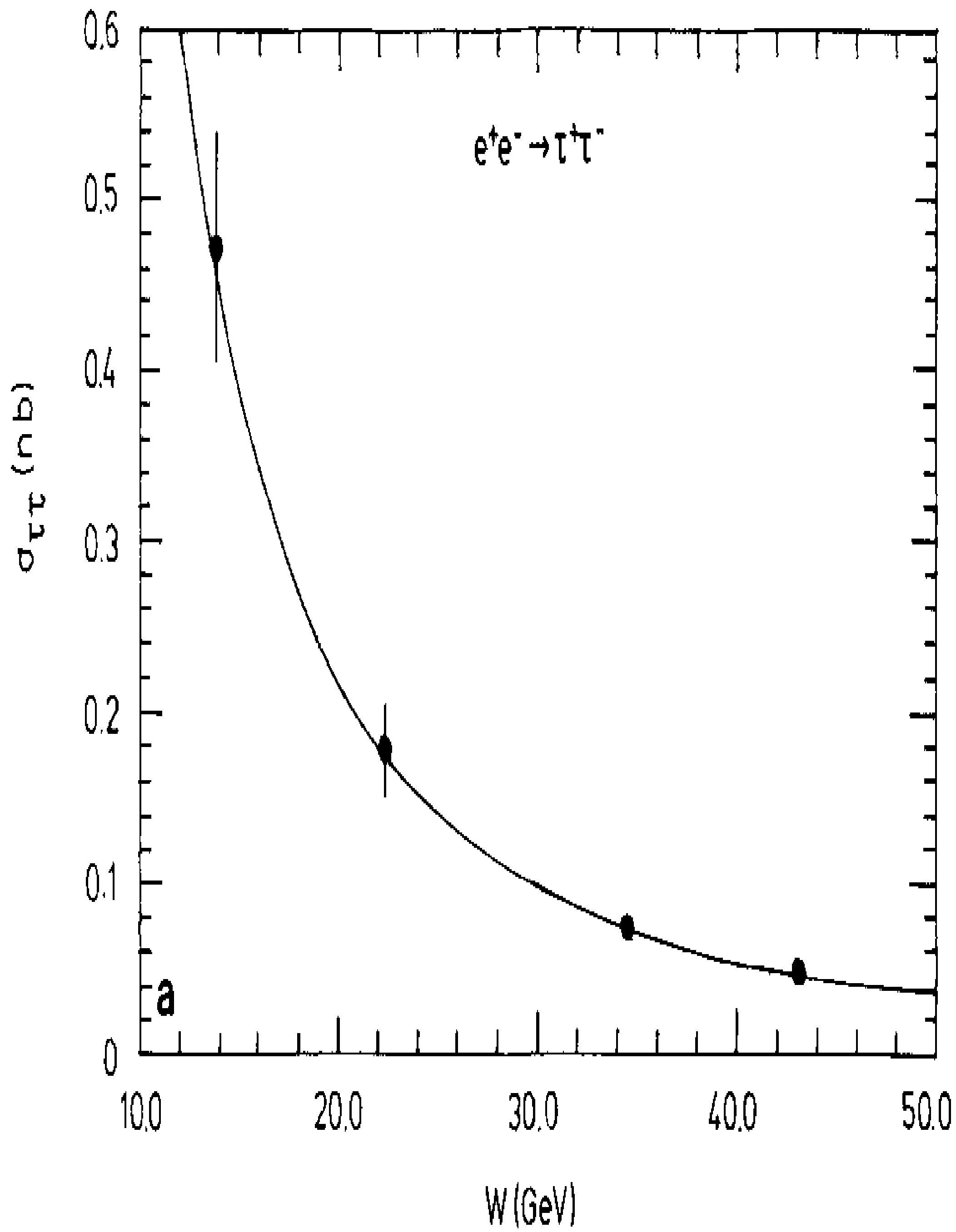

**Fig. 2.** The measured tau pair differential cross section at $W = 34.5$ GeV. The dashed line has the form $(1+\cos^2\theta)$ expected from lowest order QED, normalised to the data and the full line is the result of the fit

s section expression $$\frac{d\sigma}{d\cos\theta d\phi} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\langle|\mathcal{M}|^2\rangle}{4E^2}\frac{|p_f|}{|p_i|}$$

$$|p_f| \sim |p_i|$$

$$\frac{d\sigma}{d\cos\theta d\phi} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{g_e^4}{4E^2}\left[1+\cos^2\theta\right]$$

# Cross Section at High Energy:




- If we use our same approximation: $Mc^2 << E$

$$\sigma = \frac{\pi}{3}\left(\frac{\hbar c\alpha}{E}\right)^2 \sqrt{1-(Mc^2/E)^2}\left[1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right]$$

- becomes:

$$\sigma = \frac{\pi}{3}\left(\frac{\hbar c\alpha}{E}\right)^2$$