Lecture 15: Spin-averaged cross sections

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Last time:

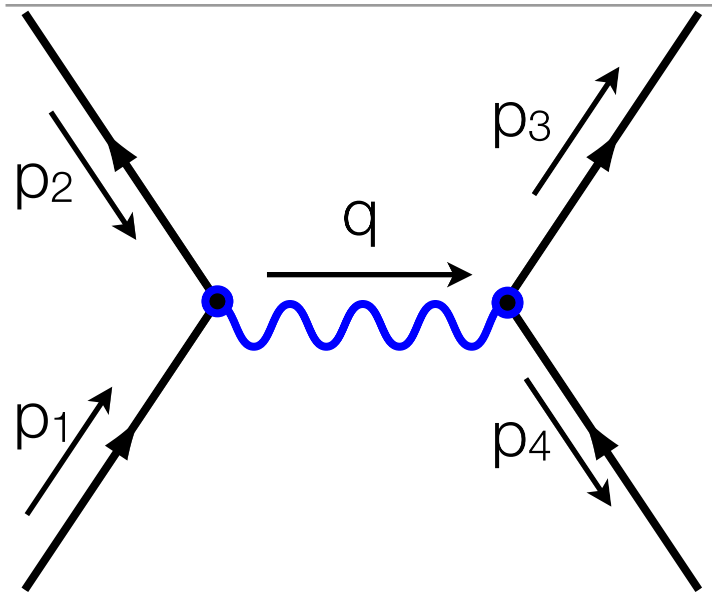
- We can calculate the amplitude *M* in QED using the Feynman rules.
- What do we do with this?
  - We want a cross section or rate, so we incorporate it with the phase space
  - For this we want $|M|^2 = M\, M^*$
- Note that the amplitudes we calculated specified the spins/helicities of the incoming and out-coming particles:
  - often the incoming particles will be unpolarized:
    - this suggests we should average over incident polarizations
      - i.e sum over possible polarizations and then divide by number of configurations
  - often we cannot measure the polarization of the out coming particles
    - this suggests we should sum over outgoing polarizations
  - we will see that this often leads to very clean expressions . . .

Recall the “completeness” relations

$$\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\mu p_\mu + mc) \qquad \sum_{s=1,2} v^s \bar{v}^s = (\gamma^\mu p_\mu - mc)$$

- This “outer” product results in a 4x4 Dirac matrix
- Note that this is a relation that involves summing over spins, just what we need for averaging/summing.
- This will play a critical role in what follows

Example diagram and amplitude:



$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\ \gamma^\mu\ v(4)]\,[\bar{v}(2)\ \gamma_\mu\ u(1)]$$

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{g_e^4}{(p_1+p_2)^4}\,[\bar{u}_3\gamma^\mu v_4]\,[\bar{v}_2\gamma_\mu u_1] \quad\times\quad [\bar{u}_3\gamma^\nu v_4]^*\,[\bar{v}_2\gamma_\nu u_1]^*$$

- Generically, we will encounter expressions like:

$$[\bar{u}_a\Gamma_1 u_b][\bar{u}_a\Gamma_2 u_b]^*$$

$$[\bar{u}_a\Gamma_1 v_b][\bar{u}_a\Gamma_2 v_b]^*$$

$$[\bar{v}_a\Gamma_1 u_b][\bar{v}_b\Gamma_2 u_b]^*$$

$$[\bar{v}_a\Gamma_1 v_b][\bar{v}_a\Gamma_2 v_b]^*$$

where Γ is some combination of γ matrices

$$[\bar{u}_a\Gamma_2 u_b]^* = [\bar{u}_a\Gamma_2 u_b]^\dagger$$

$$= [u_a^\dagger\gamma^0\Gamma_2 u_b]^\dagger = [u_b^\dagger\Gamma_2^\dagger\gamma^{0\dagger}u_a]$$

$\gamma^0\gamma^0 = 1$

$\gamma^{0\dagger} = \gamma^0$

$$= [u_b^\dagger\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u_a]$$

$\bar{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0$

$$= [\bar{u}_b\bar{\Gamma}_2 u_a]$$

Complex conjugation:

$$[\bar{u}_a \Gamma_1 u_b][\bar{u}_a \Gamma_2 u_b]^* = [\bar{u}_a \Gamma_2 u_b][\bar{u}_b \bar{\Gamma}_2 u_a]$$

- Now recall that we want to sum over the spins of the a and b particles
  - start with the b particle $\displaystyle\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\mu p_\mu + mc)$

$$\sum_{s_b} [\bar{u}_a \Gamma_2 u_b \bar{u}_b \bar{\Gamma}_2 u_a] \Rightarrow \bar{u}_a \Gamma_2 (\not{p}_b + m_b c) \bar{\Gamma}_2 u_a]$$

  - now with the a particle, let’s right out the expression with indices

$$\sum_i \sum_j [\bar{u}_a]_i \; [\Gamma_2 (\not{p}_b + m_b c) \bar{\Gamma}_2]_{ij} \; [u_a]_j$$

  - rearrange and introduce the spin summation on a

$$\sum_i \sum_j \; [\Gamma_2 (\not{p}_b + m_b c) \bar{\Gamma}_2]_{ij} \; \sum_{s_a} [u_a]_j [\bar{u}_a]_i$$

Final states

$$\sum_i \boxed{\sum_j [\bar{u}_a]_i \; [\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2]_{ij}} \; [u_a]_j$$

$$\downarrow$$

$$\boxed{[\not{p}_a + m_a c]_{ji}}$$

sum over j multiplies
the two matrices

$$\sum_i \; [\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]_{ii}$$

$$\mathrm{Tr}[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]$$

- For this, we assumed that both a and b particles were “particles” described by u spinors

- If we have antiparticles (v spinors) the “+” in the mass term becomes “-“

Contracting Lorentz Indices

- There are a number of γ matrix relations which we will derive now
  - There are a lot of them, but they are all we need
  - Get through them once (some in HW), we can just apply the results
- Start with some Lorentz "contraction" relations:

$$g^{\mu\nu} g_{\mu\nu} = 4$$

  - What does this mean? We sum 16 numbers resulting from the product of corresponding entries in $g^{\mu\nu}$ and $g_{\mu\nu}$. All are zero except for $\mu=\nu$. In those cases, we get either (1x1) or (-1x-1) so in the end, we get 4.

- Now another simple relation using the fundamental relation for γ matrices:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\not{a}\ \not{b} + \not{b}\ \not{a} = a_\mu \gamma^\mu b_\nu \gamma^\nu + b_\nu \gamma^\nu a_\mu \gamma^\mu = a_\mu b_\nu \{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\} = 2a \cdot b$$

Other relations:

- Other relations mostly result from the previous ones:

$$\gamma_\mu \gamma^\mu \qquad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$g_{\mu\nu}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = 2g_{\mu\nu} g^{\mu\nu} \qquad \gamma_\nu \gamma^\nu + \gamma_\mu \gamma^\mu = 2g_{\mu\nu} g^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = 4$$

- You can do this explicitly: $\gamma_\mu \gamma^\mu = (\gamma^0)^2 - (\gamma^1)^2 - (\gamma^2)^2 - (\gamma^3)^2$
  - but this method will run out of steam in more complicated examples
- A few more:

$$\gamma_\mu \gamma^\nu \gamma^\mu \qquad \gamma_\mu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) \qquad 2\gamma^\nu - 4\gamma^\nu \qquad -2\gamma^\nu$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu \qquad \gamma_\mu \gamma^\nu (2g^{\lambda\mu} - \gamma^\mu \gamma^\lambda) \qquad 2\gamma^\lambda \gamma^\nu - \gamma_\mu \gamma^\nu \gamma^\mu \gamma^\lambda$$

$$2\gamma^\lambda \gamma^\nu + 2\gamma^\nu \gamma^\lambda \qquad 4g^{\nu\lambda}$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu \qquad -2\gamma^\sigma \gamma^\lambda \gamma^\nu$$

Trace Relations

Tr(A+B) = Tr A + Tr B
Tr(αA) = αTr A
Tr(AB) = Tr (BA)

- Reminder: we're in a four dimensional spinor space!
  - "1" = 4x4 identity matrix $\Rightarrow$ Tr 1 = 4
- $\mathrm{Tr}(\gamma^\mu\gamma^\nu)$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \mathrm{Tr}(2g^{\mu\nu})$$
$$2 \times \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2 \times \mathrm{Tr}(g^{\mu\nu})$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4 \times g^{\mu\nu}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

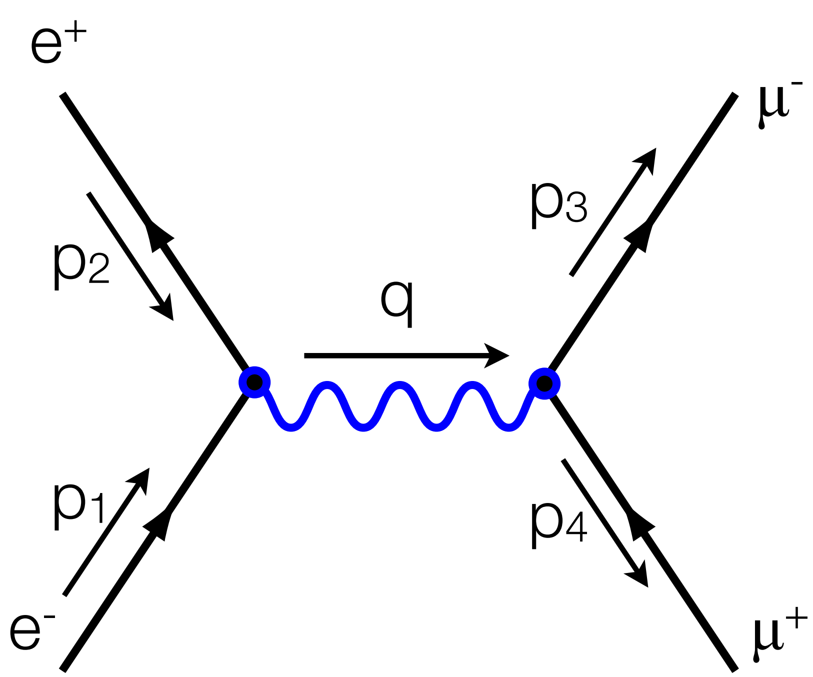
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4 \times (g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

- These relations imply:
  - $\mathrm{Tr}(\not{a}\ \not{b}) = 4a \cdot b$
  - $\mathrm{Tr}(\not{a}\ \not{b}\ \not{c}\ \not{d}) = 4\left[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\right]$
- Next: the trace of any single γ matrix is 0: $(\gamma^5)^2 = 1$ $\quad \{\gamma^\mu, \gamma^5\} = 0$
  - $\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5) = -\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu)$
  - Likewise, we can show that the trace of a product of an odd number of γ matrices is 0

Example: $e^+ + e^- \rightarrow \mu^+ + \mu^-$

- Calculate the spin-averaged amplitude
    - for simplicity let's assume that the energies are high enough that we can neglect the masses of the e and μ.
    - initial spin averaged, final spin summed.
- How do we calculate the cross section in the CM frame?

Step I/II: The Feynman Diagram and rules



$$\frac{1}{(2\pi)^4}\int d^4q \;\; \frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}(3)\; ig_e\gamma^\mu\; v(4) \quad (2\pi)^4\delta^4(q - p_3 - p_4)$$

$$\bar{v}(2)\; ig_e\gamma^\nu\; u(1) \quad (2\pi)^4\delta^4(p_1 + p_2 - q)$$

$$[\bar{u}(3)\;\gamma^\mu\; v(4)]\;\; g_{\mu\nu}\;\; [\bar{v}(2)\;\gamma^\nu\; u(1)]$$

$$i(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) \times \frac{g_e^2}{(p_1 + p_2)^2}$$

$$\mathcal{M} = -\frac{g_e^2}{(p_1 + p_2)^2}\,[\bar{u}(3)\;\gamma^\mu\; v(4)]\,[\bar{v}(2)\;\gamma_\mu\; u(1)]$$

Step III: Summing over spins:

- To get $|M|^2$ we need to take the complex conjugate of the M:

$$\mathcal{M} = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\;\gamma^\mu\;v(4)]\,[\bar{v}(2)\;\gamma_\mu\;u(1)]$$

$$\mathcal{M}^* = -\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\gamma^\nu v(4)]^*\,[\bar{v}(2)\gamma_\nu u(1)]^*$$

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4}\,[\bar{u}(3)\gamma^\mu v(4)]\,[\bar{u}(3)\gamma^\nu v(4)]^*\,[\bar{v}(2)\gamma_\mu u(1)]\,[\bar{v}(2)\gamma_\nu u(1)]^*$$

$$\sum_{\text{spins}}\;[\bar{u}(3)\gamma^\mu v(4)]\,[\bar{u}(3)\gamma^\nu v(4)]^* = \mathrm{Tr}\,[(\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)]$$

$$\sum_{\text{spins}}\;[\bar{v}(2)\gamma^\mu u(1)]\,[\bar{v}(2)\gamma^\nu u(1)]^* = \mathrm{Tr}\,[\gamma_\mu(\not{p}_1 + mc)\gamma_\nu(\not{p}_2 - mc)]$$

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4}\mathrm{Tr}\,[\gamma^\mu(\not{p}_4)\gamma^\nu(\not{p}_3)]\,\mathrm{Tr}\,[\gamma_\mu(\not{p}_1\gamma_\nu(\not{p}_2)]$$

Step IV:

- An apply the trace relations

$$\mathrm{Tr}\left[\gamma^{\mu}\not{p}_4\gamma^{\nu}\not{p}_3\right] = 4\times\left[p_4^{\mu}p_3^{\nu} + p_3^{\mu}p_4^{\nu} - g^{\mu\nu}(p_4\cdot p_3)\right]$$

$$\mathrm{Tr}\left[\gamma_{\mu}\not{p}_1\gamma_{\nu}\not{p}_2\right] = 4\times\left[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - (p_1\cdot p_2)g_{\mu\nu}\right]$$

- Carry out the contraction between the Lorentz indices:

$$\begin{aligned}16\times(\\ (p_4\cdot p_1)(p_3\cdot p_2) + (p_4\cdot p_2)(p_3\cdot p_1) - (p_1\cdot p_2)(p_4\cdot p_3)\\ (p_3\cdot p_1)(p_4\cdot p_2) + (p_3\cdot p_2)(p_4\cdot p_1) - (p_1\cdot p_2)(p_4\cdot p_3)\\ -(p_1\cdot p_2)(p_4\cdot p_3) - (p_2\cdot p_1)(p_4\cdot p_3) + 4(p_4\cdot p_3)(p_1\cdot p_2))\end{aligned}$$

$$16\times\left[2(p_1\cdot p_4)(p_2\cdot p_3) + 2(p_1\cdot p_3)(p_2\cdot p_4)\right]$$

Step IV (continued)

- Put it all together:

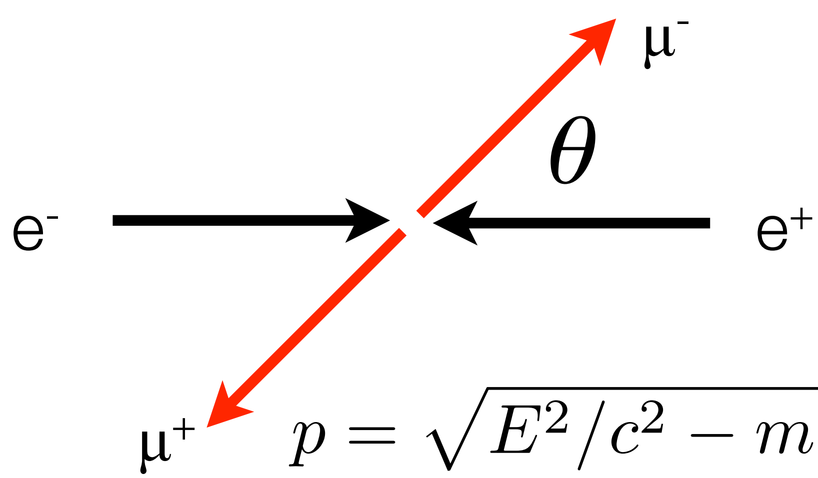
$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4} \mathrm{Tr}\left[\gamma^\mu (\not{p}_4) \gamma^\nu (\not{p}_3)\right] \mathrm{Tr}\left[\gamma_\mu (\not{p}_1) \gamma_\nu (\not{p}_2)\right]$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_e^4}{(p_1+p_2)^4} 32 \times \left[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)\right]$$

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{(p_1+p_2)^4} 8 \times \left[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)\right]$$

Step V: The Kinematics in CM:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{(p_1+p_2)^4} 8 \times [(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4)]$$



$$p = \sqrt{E^2/c^2 - m^2c^2}$$

$$p = \sqrt{E^2/c^2 - m^2c^4}$$

$$p' = \sqrt{E^2/c^2 - M^2c^2}$$

$$p_1 = (E/c, 0, p)$$
$$p_2 = (E/c, 0, -p)$$
$$p_3 = (E/c, p' \sin\theta, p' \cos\theta)$$
$$p_4 = (E/c, -p' \sin\theta, -p' \cos\theta)$$