Photons and Quantum Electrodynamics

H. A. Tanaka

The Photon

- Apart from $\bar{\psi}\psi$ we need some other particle/object with definite Lorentz transformation properties to make Lorentz invariants
  - What would we do with the "vector" term $\bar{\psi}\gamma^{\mu}\psi$ to get a Lorentz scalar?
- Recall the photon:
  - Classically, we have Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{J}$$

  - Recall that we can re-express the Maxwell equations using potentials:

$$\mathbf{E} = -\nabla\phi \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

  - these can in turn be combined to make a 4 vector: $A^{\mu} = (\phi, \mathbf{A})$
  - Likewise for the "source" terms ρ and **J**: $J^{\mu} = (c\rho, \mathbf{J})$

Maxwell's Equation in Lorentz Covariant Form

- All four equations can be expressed as:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu =$$

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- The issue is that A is (far) from unique:
  - Consider: $A_\mu \to A_\mu + \partial_\mu \lambda$

$$\partial_\mu \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu(\partial_\mu (A^\mu + \partial^\mu \lambda) =$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu (A^\mu) + \partial_\mu \partial^\mu \partial^\nu \lambda - \partial^\nu \partial_\mu \partial^\mu \lambda$$

  - the last terms cancel, so the "new" A_μ is also a solution to Maxwell's solution
  - they are physically the same, so we can make some conventions:
- "Lorentz gauge condition": $\partial_\mu A^\mu = 0$ $\quad \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$
- "Coulomb gauge" $A^0 = 0$ $\quad \nabla \cdot \mathbf{A} = 0$

Solutions to the Maxwell Equation in Free Space:

- "Free" means no sources (charges, currents): $J^\mu=0$

$$\partial^\mu \partial_\mu A^\nu = 0$$

- Find solution as usual by ansatz:

$$A^\mu(x) = a\ e^{-ip\cdot x}\epsilon^\mu(p)$$

- Now check:

$$\partial_\mu A^\nu(x) = -ip_\mu\ a\ e^{-ip\cdot x}\epsilon^\nu(p)$$

$$\partial^\mu \partial_\mu A^\nu(x) = (-i)^2 p^\mu p_\mu\ a\ e^{-ip\cdot x}\epsilon^\nu(p) = 0$$

$$\partial_\mu A^\mu = 0 \Rightarrow p_\mu \epsilon^\mu(p) = 0$$

$$p^2 = m^2c^2 = 0$$

$$A^0 = 0 \Rightarrow \epsilon^0 = 0$$

$$\Rightarrow \mathbf{p}\cdot\epsilon = 0$$

- Conclusions:
  - Photon is massless
  - Polarization ε is transverse to photon direction:
    - it has two degrees of freedom/polarizations

Making a "scalar" object:

- In the end, these spaces must collapse:
  - In Lorentz space, this happens by contracting indices: $g_{\mu\nu} a^\mu b^\nu = a^\mu b_\mu$
  - In spinor space, products of adjoint spinors with spinors (with gamma matrices possibly in between): $\bar{u_1} \Gamma v_2$ Γ=(product of g matrices)
- but some expressions have structure in both:

sum over μ collapses the Lorentz structure

$$\mathcal{M} = -\frac{g_e^2}{(p_1 + p_2)^2} [\bar{u}(3)\ \gamma^\mu\ v(4)]\ [\bar{v}(2)\ \gamma_\mu\ u(1)]$$

Contracted in spinor space, but not in Lorentz

Same here

Product of 4x4 matrices in spinor space

$$\gamma^\nu = S^{-1} \gamma^\mu S \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

sum over μ collapses the Lorentz structure

Reminder of Dirac Spinors

- We can now construct the column vector u:

Use "positive" energy solutions

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ p_z c/(E+mc^2) \\ (p_x + ip_y)c/(E+mc^2) \end{pmatrix} \qquad u_2 = N \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \end{pmatrix}$$

electrons

$$-v_2 \equiv u_3 = N \begin{pmatrix} p_z c/(E+mc^2) \\ (p_x + ip_y)c/(E+mc^2) \\ 1 \\ 0 \end{pmatrix} \qquad v_1 \equiv u_4 = N \begin{pmatrix} (p_x - ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \\ 0 \\ 1 \end{pmatrix}$$

Use "negative" energy solutions

positrons

A second look at Dirac spinors

electrons

$$\psi(x) = ae^{-(i/\hbar)p\cdot x}\ u^s(p)$$

positrons

$$\psi(x) = ae^{(i/\hbar)p\cdot x}\ v^s(p)$$

- "s" labels the spin states (two for electrons/positrons)
- The exponential term sets the space/time = energy/momentum
- Let's look at the "spinor" part u,v which determines the "Dirac structure":
  - If we insert ψ into the Dirac equation, we get:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu - mc)u^s(p) = 0$$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \Rightarrow (-\gamma^\mu p_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu + mc)v^s(p) = 0$$

"momentum space Dirac equations"

- If we take the adjoint of these equations, we get:

$$\bar{u}^s(\gamma^\mu p_\mu - mc) = 0 \qquad \bar{v}^s(\gamma^\mu p_\mu + mc) = 0$$

Orthogonality and Completeness of Spinors:

- From the explicit form of our u/v spinors, we can also show:

$$\bar{u}^i u^j = 2mc\,\delta^{ij} \qquad \bar{v}^i v^j = -2mc\,\delta^{ij} \qquad \bar{u}^i v^j = \bar{v}^i u^j = 0$$

- We can also show:

$$\sum_{s=1,2} u^s \bar{u}^s = (\gamma^\mu p_\mu + mc) \qquad \sum_{s=1,2} v^s \bar{v}^s = (\gamma^\mu p_\mu - mc)$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} (b_1, b_2, b_3, b_4) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \\ a_4 b_1 & a_4 b_2 & a_4 b_3 & a_4 b_4 \end{pmatrix}$$

Photon Polarizations and Orthogonality:

- We showed that the polarization 4-vector ε^μ with the Lorentz and Coloumb gauge conditions must satisfy:

$$\mathbf{p} \cdot \epsilon = 0$$

- We noted that this allows two degrees of freedom corresponding to transversely polarized electromagnetic fields.
  - We need to two orthogonal ε basis vectors to span the space
- For example, if the photon is moving in the z direction, we can choose:

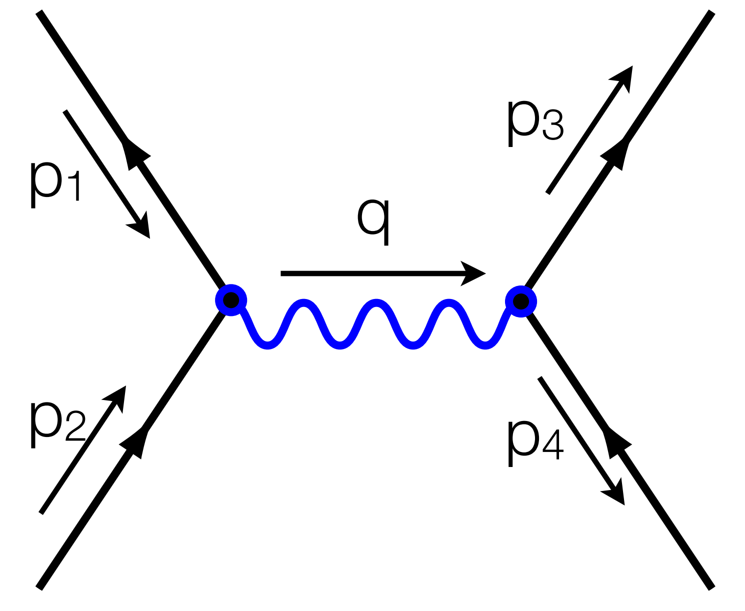
$$\epsilon^1_\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon^2_\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- The polarization vectors satisfy orthogonality/completeness relations:

$$\epsilon^{i*}_\mu \epsilon^{\mu j} = -\delta^{ij} \qquad \sum_{s=1,2} \epsilon^s_i \epsilon^{s*}_j = \delta_{ij} - \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j$$

The Feynman Rules: External Lines

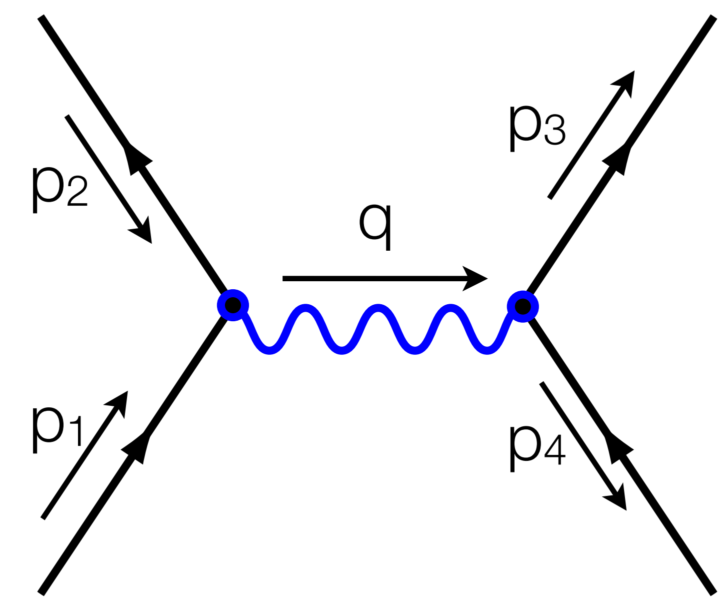
- First right down the Feynman diagram(s) for the process and label the momentum flow
  - use p's for external lines, q's for internal (Griffiths convention).
  - Note that there are two flows:
    - "particle/antiparticle"
    - momentum
    - These are separate



- Now the components of the expression
  - External Lines:
    - Electrons: incoming $u^s(p)$ outgoing $\bar{u}^s(p)$
    - Positrons: incoming $\bar{v}^s(p)$ outgoing $v^s(p)$
    - Photons: incoming $\epsilon_\mu(p)$ outgoing $\epsilon^*_\mu(p)$

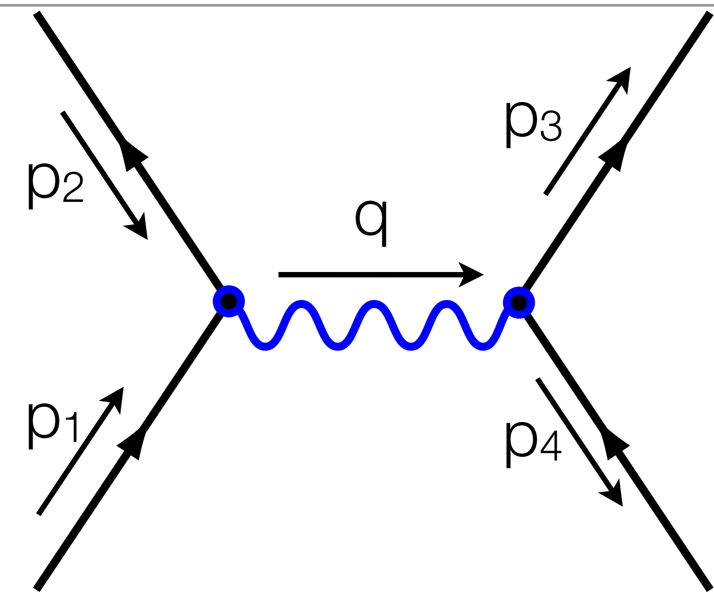
The Feynman Rules: Vertices and Propagators:

- For each QED vertex: $ig_e\gamma^\mu \ (2\pi)^4\delta^4(k_1+k_2+k_3)$
  - where as before, momentum is "+" incoming, "-" outgoing from vertex
- Internal lines:
  - electron/positron propagator $\dfrac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$
  - Photon propagator $\dfrac{-ig_{\mu\nu}}{q^2}$
    - indices match vertices/polarization
  - Integral over momentum: $\dfrac{d^4q}{(2\pi)^4}$



- Finally: cancel the overall delta function, what remains is -iM

Example:

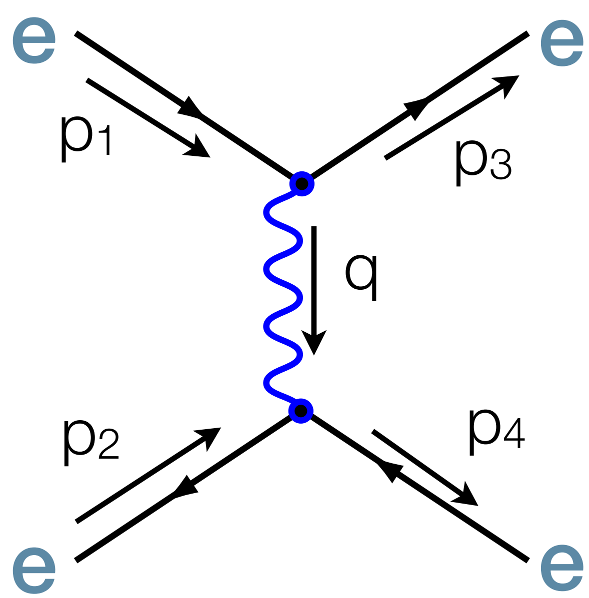


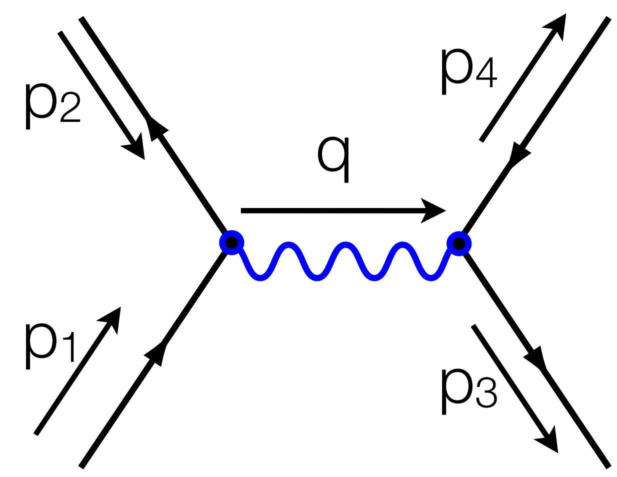
- Order matters due to Dirac matrix structure (photon part doesn't care)
- Griffiths: go backward through the fermion lines:
  - In the "final state": $\bar{u}(3)\ ig_e\gamma^\mu\ v(4) \quad (2\pi)^4\delta^4(q-p_3-p_4)$
  - In the "initial state": $\bar{v}(2)\ ig_e\gamma^\nu\ u(1) \quad (2\pi)^4\delta^4(p_1+p_2-q)$
  - Throw in the internal photon propagator: $\dfrac{1}{(2\pi)^4}\displaystyle\int d^4q \quad \frac{-ig_{\mu\nu}}{q^2}$

$$i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\times\frac{g_e^2}{(p_1+p_2)^2}\ [\bar{u}(3)\ \gamma^\mu\ v(4)]\ g_{\mu\nu}\ [\bar{v}(2)\ \gamma^\nu\ u(1)]$$

$$\mathcal{M}=-\frac{g_e^2}{(p_1+p_2)^2}\,[\bar{u}(3)\ \gamma^\mu\ v(4)]\,[\bar{v}(2)\ \gamma_\mu\ u(1)]$$

Example: $e^+ + e^- \rightarrow e^+ + e^-$





$$\bar{u}(3)\ ig_e\gamma^\mu\ u(1)\ \ \bar{v}(2)\ ig_e\gamma^\nu\ v(4)\ \ \frac{-ig_{\mu\nu}}{(p_1-p_3)^2}$$

$$\bar{u}(3)\ ig_e\gamma^\rho\ v(4)\ \ \bar{v}(2)\ ig_e\gamma^\sigma\ u(1)\ \ \frac{-ig_{\rho\sigma}}{(p_1+p_2)^2}$$

$$(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$$