Lecture 13: Properties of the Dirac Equation

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Lorentz Properties:

- The Dirac equation "works" in all reference frames.
  - What exactly does this mean?

"Lorentz Covariant"

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0$$

- i, $\hbar$, m and c are constants that don't change with reference frames.
- ∂_μ and ψ will change with reference frames, however.
  - ∂_μ is a derivative that will be taken with respect to the space-time coordinates in the new reference frame. We'll call this ∂'_μ
  - how does ψ change?
    - $\psi' = S\psi$ where ψ' is the spinor in the new reference frame
- Putting this together, we have the following transformation of the equation when evaluating it in a new reference frame

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \qquad \Rightarrow \qquad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

What properties does S need to make this work?

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

Transformation of derivatives

- Since we know how to relate space time coordinates in one reference with another (i.e. Lorentz transformation), we can do the same for the derivatives
- Using the chain rule, we get: $$\partial'_\mu \equiv \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}} \frac{\partial}{\partial x^\nu}$$
  - where we view x as a function x' (i.e. the original coordinates as a function of the transformed or primed coordinates).
  - Note the summation over ν
- if the primed coordinates moving along the x axis with velocity βc:

$$\begin{aligned} x^0 &= \gamma(x^{0\prime} + \beta x^{1\prime}) \\ x^1 &= \gamma(x^{1\prime} + \beta x^{0\prime}) \\ x^2 &= x^{2\prime} \\ x^3 &= x^{3\prime} \end{aligned}$$

$$(\nu = 0, \mu = 0) \Rightarrow \frac{\partial x^0}{\partial x^{0\prime}} = \gamma$$

$$(\nu = 0, \mu = 1) \Rightarrow \frac{\partial x^0}{\partial x^{1\prime}} = \gamma\beta$$

etc.

Transforming the Dirac Equation:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \qquad \Rightarrow \qquad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

$$i\hbar\gamma^\mu\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu(S\psi) - mc\,(S\psi) = 0$$

S is constant in space time, so we can move it to the left of the derivatives

$$i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,(S\psi) = 0$$

Now slap S^{-1} from both sides

$$i\hbar\gamma^\nu\partial_\nu\psi - mc\,\psi = 0 \qquad S^{-1} \rightarrow i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,S\psi = 0$$

Since these equations must be the same, S must satisfy

$$\gamma^\nu = S^{-1}\gamma^\mu S\ \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

Example: The parity operator

- For the parity operator, we want to invert the spatial coordinates while keeping the time coordinate unchanged:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\frac{\partial x_0}{\partial {x_0}'} = 1 \qquad \frac{\partial x_1}{\partial {x_1}'} = -1$$

$$\frac{\partial x_2}{\partial {x_2}'} = -1 \qquad \frac{\partial x_3}{\partial {x_3}'} = -1$$

- We then have

$$\gamma^0 = S^{-1}\gamma^0 S$$
$$\gamma^1 = -S^{-1}\gamma^1 S$$
$$\gamma^2 = -S^{-1}\gamma^2 S$$
$$\gamma^3 = -S^{-1}\gamma^3 S$$

Recalling

$$\gamma^\nu = S^{-1}\gamma^\mu S \, \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\gamma^0)^2 = 1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0$$

We find that γ^0 satisfies our needs

$$\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

$$\gamma^i = -\gamma^0\gamma^i\gamma^0 = \gamma^0\gamma^0\gamma^i = \gamma^i$$

$$S_P = \gamma^0$$

"Bilinear Covariants"

- We can now study the Lorentz transformation properties of Dirac fields
- Recall that for four vectors a^μ, b^μ
  - a^μ, b^μ transform as Lorentz vectors (obviously)
  - $a^\mu b_\mu$ is a scalar (does not change under Lorentz transformations
  - $a^\mu b^\nu$ is a tensor (each has a Lorentz transformation)
- From the previous discussion, we know:
  - Dirac spinors have four components, but don't transform as Lorentz vectors
  - How do combinations of Dirac spinors change under Lorentz Transformations?

How do we construct a scalar?

- We can use γ^0: define: $\bar{\psi} = \psi^\dagger \gamma^0$

homework

  - We can also show generally that $S^\dagger \gamma^0 S = \gamma^0$
  - This gives us $\bar{\psi}\psi \Rightarrow \psi^\dagger S^\dagger \gamma^0 S \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$
  - so this is a Lorentz scalar (possibly a pseudo scalar)
- We can construct the parity operator to check how $\bar{\psi}\psi$ transforms under the parity operation.
  - Recall $S_P = \gamma^0$
    - We can investigate how $\bar{\psi}\psi$ transforms under parity

$$\bar{\psi}\psi \Rightarrow (\psi^\dagger S_P^\dagger \gamma^0)(S_P \psi) = \psi^\dagger \gamma^0 \gamma^0 \gamma^0 \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$$

$\bar{\psi}\psi$ doesn't change sign under parity
it is a Lorentz scalar

The γ^5 operator

- Define the operator γ^5 as:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- It anticommutes with all the other γ matrices:

$$\{\gamma^\mu, \gamma^5\} = 0$$

  - use the canonical anti-commutation relations to move γ^μ to the other side
  - γ^μ will anti-commute with for $\mu \neq \nu$
  - γ^μ will commute when $\mu = \nu$
- We can then consider the quantity $\bar{\psi}\gamma^5\psi$
  - Show in homework that this is invariant under Loretnz transformation.
- What about under parity:

$$\bar{\psi}\gamma^5\psi \Rightarrow (\psi^\dagger S_P^\dagger)\gamma^0\gamma^5(S_P\psi) = -(\psi^\dagger S_P^\dagger)\gamma^0 S_P\gamma^5\psi = -\psi^\dagger\gamma^0\gamma^5\psi = -\bar{\psi}\gamma^5\psi$$

$\bar{\psi}\gamma^5\psi$ is a pseudoscalar

Other Combinations

- We can use γ^μ to make vectors and tensor quantities:

$\bar{\psi}\psi$	scalar	1 component	
$\bar{\psi}\gamma^5\psi$	pseudoscalar	1 component	
$\bar{\psi}\gamma^\mu\psi$	vector	4 components	
$\bar{\psi}\gamma^\mu\gamma^5\psi$	pseudovector	4 components	
$\bar{\psi}\sigma^{\mu\nu}\psi$	antisymmetric tensor	6 components	$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

- You can tell the transformation properties by looking at the Lorentz indices
  - γ^5 introduces a sign (adds a "pseudo")
  - Every combination of $\psi^*_i\psi_j$ is a linear combination of the above.
- We will see that the above are the basis for creating interactions with Dirac particles. Interactions will be classified as "vector" or "pseudovector", etc.

Angular Momentum and the Dirac Equation:

- Conservation:
  - In quantum mechanics, what is the condition for a quantity to be conserved?

$$[H, Q] = 0$$

  - Free particle Hamiltonian in non-relativistic quantum mechanics:

$$H = \frac{p^2}{2m} \qquad [H, p] = [\frac{p^2}{2m}, p] = \frac{1}{2m}[p^2, p] = 0$$

  - thus we conclude that the momentum p is conserved
- If we introduce a potential:

$$[H, p] = [\frac{p^2}{2m} + V(x), p] = \frac{1}{2m}[p^2 + V(x), p] \neq 0$$

  - thus, momentum is not conserved

Hamiltonian for the Dirac Particle:

- Starting with the Dirac equation,

$$(\gamma^\mu p_\mu - mc)\psi = 0$$

- determine the Hamiltonian by solving for the energy

- Hints:

$$(\gamma^0)^2 = 0$$

- Answer: $H = c\gamma^0 \left(\gamma \cdot \mathbf{p} + mc\right)$

Is orbital angular momentum conserved?

- We want to evaluate $[H, \vec{L}]$
- Recall: $\vec{L} = \vec{x} \times \vec{p}$ $\qquad L_i = \epsilon_{ijk} x^j p^k$

$$H = c\gamma^0 \left(\gamma \cdot \mathbf{p} + mc\right) \qquad H = c\gamma^0 \left(\delta_{ab}\gamma^a p^b + mc\right)$$

$$[H, L_i] = [c\gamma^0 \left(\delta_{ab}\gamma^a p^b + mc\right), \epsilon_{ijk} x^j p^k]$$

- which parts do not commute?

$$[c\gamma^0 \delta_{ab}\gamma^a p^b, \epsilon_{ijk} x_j p_k] \qquad [mc, \epsilon_{ijk} x_j p_k]$$

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$$c\gamma^0 \delta_{ab} \epsilon_{ijk} [p^b, x^j p^k] \qquad [A, BC] = [A, B]C + B[A, C]$$

$$c\delta_{ab}\epsilon_{ijk}\gamma^0\gamma^a(-i\hbar\delta^{bj}p^k) \qquad -i\hbar c\gamma^0 \epsilon_{ijk}\gamma^j p^k \qquad -i\hbar c\gamma^0 (\vec{\gamma} \times \vec{p})$$

Orbital angular momentum is not conserved

Spin

- Consider the operator $\vec{S} = \frac{\hbar}{2}\vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ acting on Dirac spinors
- Note that it satisfies all the properties of an angular momentum operator.
- Let's consider the commutator of this with the hamiltonian $H = c\gamma^0\left(\gamma \cdot \mathbf{p} + mc\right)$

$$\frac{\hbar c}{2}[\gamma^0\vec{\gamma} \cdot \vec{p} + \gamma^0 mc, \vec{\Sigma}]$$

- once again, consider in component/index notation

$$[H, S^i] = \frac{\hbar c}{2}[\gamma^0\delta_{ab}\gamma^a p^b + \gamma^0 mc, \Sigma^i]$$

- which part doesn't commute?

$$\frac{\hbar c}{2}\delta_{ab}p^b[\gamma^0\gamma^a, \Sigma^i] \qquad [AB, C] = [A, C]B + A[B, C]$$

$$\frac{\hbar c}{2}\delta_{ab}p^b\gamma^0[\gamma^a, \Sigma^i] \qquad [\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}]$$

$$\begin{pmatrix} 0 & [\sigma_a, \sigma_i] \\ -[\sigma^a, \sigma_i] & 0 \end{pmatrix} \qquad \epsilon_{aij}\begin{pmatrix} 0 & \sigma_j \\ -\sigma^j & 0 \end{pmatrix} \qquad \epsilon_{aij}\gamma^j$$

Putting it together:

$$\frac{\hbar c}{2}\delta_{ab}p^b\gamma^0[\gamma^a,\Sigma^i] \qquad \frac{\hbar c}{2}\delta_{ab}p^b\gamma^0\epsilon_{aij}\gamma^j \qquad \frac{\hbar c}{2}p^b\gamma^0\epsilon_{bij}\gamma^j$$

$$[H,\vec{S}] = \frac{\hbar c}{2}\gamma^0(\vec{\gamma}\times\vec{p})$$

The “total” spin operator:

- Define the operator:

$$\mathbf{S}^2 = \mathbf{S} \cdot \mathbf{S} \qquad \qquad \mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

- Calculate its eigenvalue for an arbitrary Dirac spinor:

- If the eigenvalue of the operator gives s(s+1), where s is the spin, what is the spin of a Dirac particle?

The Photon

- Apart from $\bar{\psi}\psi$ we need some other particle/object with definite Lorentz transformation properties to make Lorentz invariants
  - We will learn later that $\bar{\psi}\psi$ will make the mass term for the particle.
  - What would we do with the "vector" term $\bar{\psi}\gamma^{\mu}\psi$ to get a Lorentz scalar?
- Recall the photon:
  - Classically, we have Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{J}$$

  - Recall that we can re-express the Maxwell equations using potentials:

$$\mathbf{E} = -\nabla\phi \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

  - these can in turn be combined to make a 4 vector: $A^{\mu} = (\phi, \mathbf{A})$
  - Likewise for the "source" terms ρ and **J**: $J^{\mu} = (c\rho, \mathbf{J})$

Maxwell's Equation in Lorentz Covariant Form

- All four equations can be expressed as:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu =$$

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- The issue is that A is (far) from unique:
  - Consider: $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$

$$\partial_\mu \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu(\partial_\mu (A^\mu + \partial^\mu \lambda) =$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu (A^\mu) + \partial_\mu \partial^\mu \partial^\nu \lambda - \partial^\nu \partial_\mu \partial^\mu \lambda$$

  - the last term is zero, so this new A_μ is also a solution to Maxwell's solution
  - they are physically the same, so we can make some conventions:
- "Lorentz gauge condition": $\partial_\mu A^\mu = 0$ $\qquad \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$
- "Coulomb gauge" $\qquad A^0 = 0 \qquad \nabla \cdot \mathbf{A} = 0$

Solutions to the Maxwell Equation in Free Space:

- "Free" means no sources (charges, currents): $J^\mu=0$ $\quad \partial^\mu \partial_\mu A^\nu = 0$
- Find solution as usual by guessing:

$$A^\mu(x) = a\ e^{-ip\cdot x}\epsilon^\mu(p)$$

- Now check:

$$\partial_\mu A^\nu(x) = -ip_\mu\ a\ e^{-ip\cdot x}\epsilon^\nu(p)$$

$$\partial^\mu \partial_\mu A^\nu(x) = (-i)^2 p^\mu p_\mu\ a\ e^{-ip\cdot x}\epsilon^\nu(p) = 0$$

$$\partial_\mu A^\mu = 0 \Rightarrow p_\mu \epsilon^\mu(p) = 0$$

$$p^2 = m^2c^2 = 0$$

$$A^0 = 0 \Rightarrow \epsilon^0 = 0$$

$$\Rightarrow \mathbf{p}\cdot\epsilon = 0$$

- Conclusions:
  - Photon is massless
  - Polarization ε is transverse to photon direction:
    - it has two degrees of freedom/polarizations

A second look at Dirac spinors

electrons

$$\psi(x) = ae^{-(i/\hbar)p\cdot x}\ u^s(p)$$

positrons

$$\psi(x) = ae^{(i/\hbar)p\cdot x}\ v^s(p)$$

- "s" labels the spin states (remember that there were two for electrons/ positrons)
- The exponential term sets the space/time = energy/momentum
- Let's look at the "spinor" part u,v which determines the "Dirac structure":
  - If we insert ψ into the Dirac equation, we get:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu - mc)u^s(p) = 0$$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \Rightarrow (-\gamma^\mu p_\mu - mc)\psi = 0 \Rightarrow (\gamma^\mu p_\mu + mc)v^s(p) = 0$$

sometimes called "momentum space Dirac equations"

  - If we take the adjoint of these equations, we get:

$$\bar{u}^s(\gamma^\mu p_\mu - mc) = 0 \qquad\qquad \bar{v}^s(\gamma^\mu p_\mu + mc) = 0$$