The Dirac Equation

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Relativistic Wave Equations:

- In non-relativistic quantum mechanics, we have the Schrödinger Equation:

$$\mathbf{H}\psi = i\hbar\frac{\partial}{\partial t}\psi \qquad \mathbf{H} = \frac{\mathbf{p}^2}{2m} \qquad \mathbf{p} \Leftrightarrow -i\hbar\nabla$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$$

- Inspired by this, Klein and Gordon (and actually Schrödinger) tried:

$$E^2 = p^2c^2 + m^2c^4 = c^2(-\hbar^2\nabla^2 + m^2c^2)\psi = -\hbar^2\frac{\partial^2}{\partial t^2}\psi$$

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\psi = \frac{m^2c^2}{\hbar^2}\psi$$

$$\partial_\mu = (\partial_0,\ \partial_1,\ \partial_2,\ \partial_3) = \left(\frac{\partial}{\partial ct},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) \qquad (-\hbar^2\partial^\mu\partial_\mu + m^2c^2)\psi = 0$$

"Manifestly Lorentz Invariant"

Issues with KG and Dirac:

- Within the context of quantum mechanics, this had some issues:
  - As it turns out, this allows negative probability densities: $|\psi|^2 < 0$
  - Dirac traced this to the fact that we had second-order time derivative
    - "factor" the E/p relation to get linear relations and obtained:

$$p_\mu p^\mu - m^2c^2 = 0 \Rightarrow (\alpha^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

- and found that:

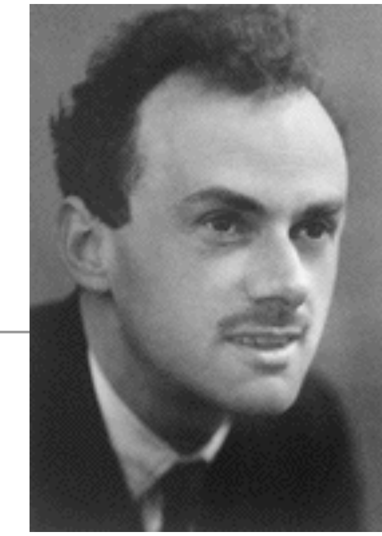
$$\alpha^\kappa = \gamma^\kappa$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- Dirac found that these relationships could be held by matrices, and that the corresponding wave function must be a "vector".

$$\gamma^\mu p_\mu - mc = 0 \Rightarrow (i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

The Dirac Equation in its many forms:



$$(i\hbar\,\partial\!\!\!/ - mc)\psi = 0 \qquad a\!\!\!/ \equiv a_\mu \gamma^\mu$$

$$(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0 \qquad a\!\!\!/ \equiv a_\mu \gamma^\mu = a_0\gamma^0 - a_1\gamma^1 - a_2\gamma^2 - a_3\gamma^3$$

$$\partial_\mu = (\partial_0,\ \partial_1,\ \partial_2,\ \partial_3) = \left(\frac{\partial}{\partial ct},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)$$

$$\left[i\hbar(\gamma^0\partial_0 - \gamma^1\partial_1 - \gamma^2\partial_2 - \gamma^3\partial_3) - mc\right]\psi = 0$$

$$\left[i\hbar\left(\gamma^0\frac{\partial}{\partial ct} - \gamma^1\frac{\partial}{\partial x} - \gamma^2\frac{\partial}{\partial y} - \gamma^3\frac{\partial}{\partial z}\right) - mc\right]\psi = 0$$

Now the "gamma" Matrices:

$$\gamma^\mu = (\gamma^0,\ \gamma^1,\ \gamma^2, \gamma^3) \qquad \vec{\sigma} = (\sigma^1,\ \sigma^2,\ \sigma^3) = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

- Note that this is a particular representation of the matrices
- Any set of matrices satisfying the anti-commutation relations works
  $$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$
- There are an infinite number of possibilities: this particular one (Björken-Drell) is just one example

In full glory:

$$\left[i\hbar \begin{pmatrix} \frac{\partial}{\partial ct} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial ct} & -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & -\frac{\partial}{\partial ct} & 0 \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial ct} \end{pmatrix} - \begin{pmatrix} mc & 0 & 0 & 0 \\ 0 & mc & 0 & 0 \\ 0 & 0 & mc & 0 \\ 0 & 0 & 0 & mc \end{pmatrix} \right] \begin{pmatrix} \overset{\psi_A}{\boxed{\begin{matrix} \psi_1 \\ \psi_2 \end{matrix}}} \\ \underset{\psi_B}{\boxed{\begin{matrix} \psi_3 \\ \psi_4 \end{matrix}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p_\mu \;\Leftrightarrow\; i\hbar\partial_\mu$$

$$\begin{pmatrix} p_0 - mc & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -p_0 - mc \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

Consider applying another matrix to this equation

$$\begin{pmatrix} p_0 + mc & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -p_0 + mc \end{pmatrix} \begin{pmatrix} p_0 - mc & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -p_0 - mc \end{pmatrix} = \begin{pmatrix} p_0^2 - m^2c^2 - (\mathbf{p}\cdot\sigma)^2 & 0 \\ 0 & p_0^2 - m^2c^2 - (\mathbf{p}\cdot\sigma)^2 \end{pmatrix}$$

From problem 4.20c

$$(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b})$$

$$(\sigma\cdot\mathbf{p})^2 = \mathbf{p}\cdot\mathbf{p}$$

$$\begin{pmatrix} p_0^2 - \mathbf{p}^2 - m^2c^2 & 0 \\ 0 & p_0^2 - \mathbf{p}^2 - m^2c^2 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0$$

- But this is just the KG equation four times
    - Wavefunctions that satisfy the Dirac equation also satisfy KG

Solutions to the Dirac Equation:

- Consider a particle at rest: $\psi(x) \sim e^{-ik\cdot x} = e^{\frac{-i}{\hbar}(\frac{E}{c}t - \mathbf{p}\cdot\mathbf{x})}$

$$k^\mu = \frac{1}{\hbar}\,(E/c,\ p_x,\ p_y,\ p_z)$$

  - Particle has no spatial dependence, only time dependence.

$$(i\hbar\gamma^0 \frac{\partial}{\partial ct} - mc)\psi = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t}\psi_A \\ \frac{\partial}{\partial t}\psi_B \end{pmatrix} = \frac{-imc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

  - Note that the equation breaks up into two independent parts:

$$\frac{\partial}{\partial t}\psi_A = -i\frac{mc^2}{\hbar}\psi_A \qquad -\frac{\partial}{\partial t}\psi_B = -i\frac{mc^2}{\hbar}\psi_B$$

$$\psi_A(t) = e^{-i(\frac{mc^2}{\hbar})t}\psi_A(0) \qquad \psi_B(t) = e^{-i(-\frac{mc^2}{\hbar})t}\psi_B(0)$$

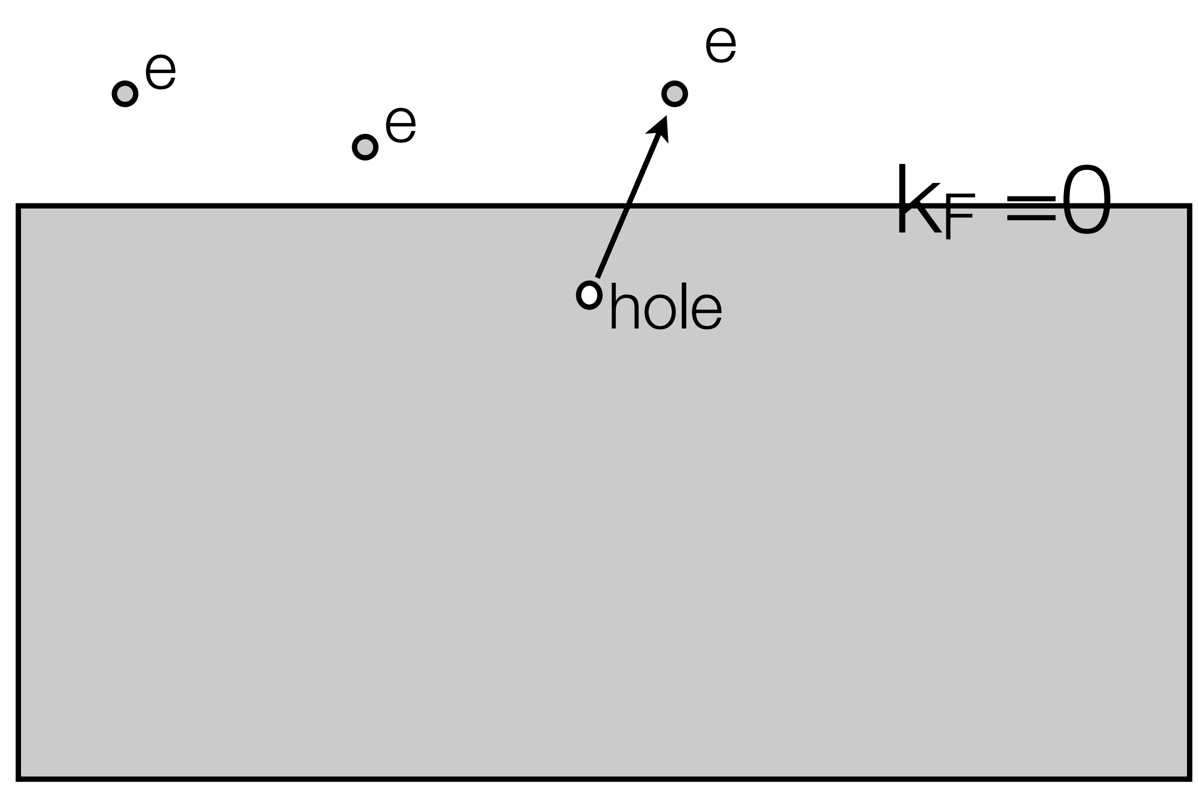
Dirac's Dilemma:

- ψ_B appears to have negative energy

$$\psi_A(t) = e^{-i(\frac{mc^2}{\hbar})t}\psi_A(0) \qquad\qquad \psi_B(t) = e^{-i(-\frac{mc^2}{\hbar})t}\psi_B(0)$$

- Why don't all particles fall down into these states (and down to -∞)?
- Dirac's excuse: all electron states in the universe up to a certain level (say E=0) are filled.
- Pauli exclusion prevents collapse of states down to E = -∞
- We can "excite" particles out of the sea into free states

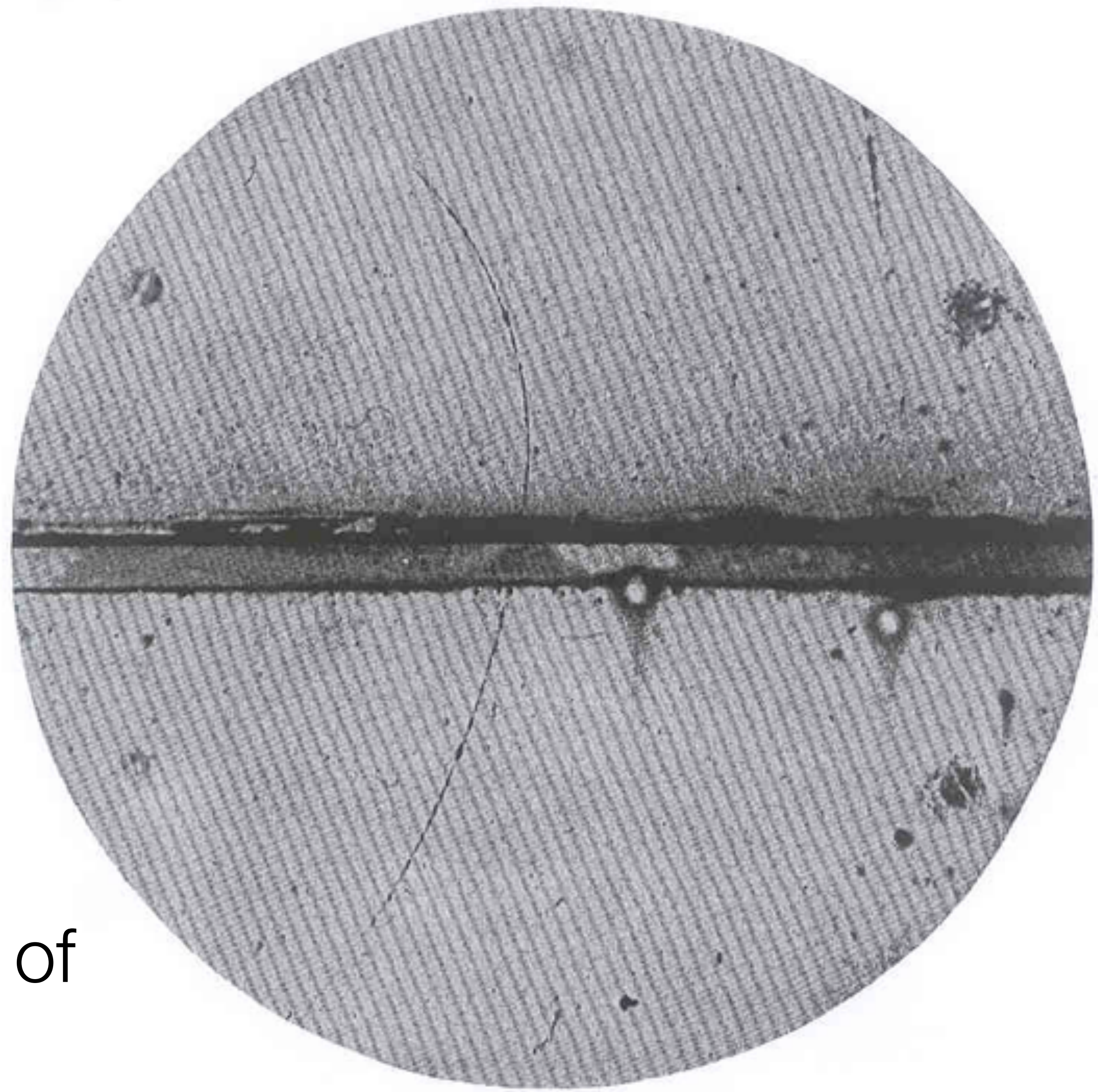
  This leaves a "hole" that looks like a particle with opposite properties (positive charge, opposite spin, etc.)



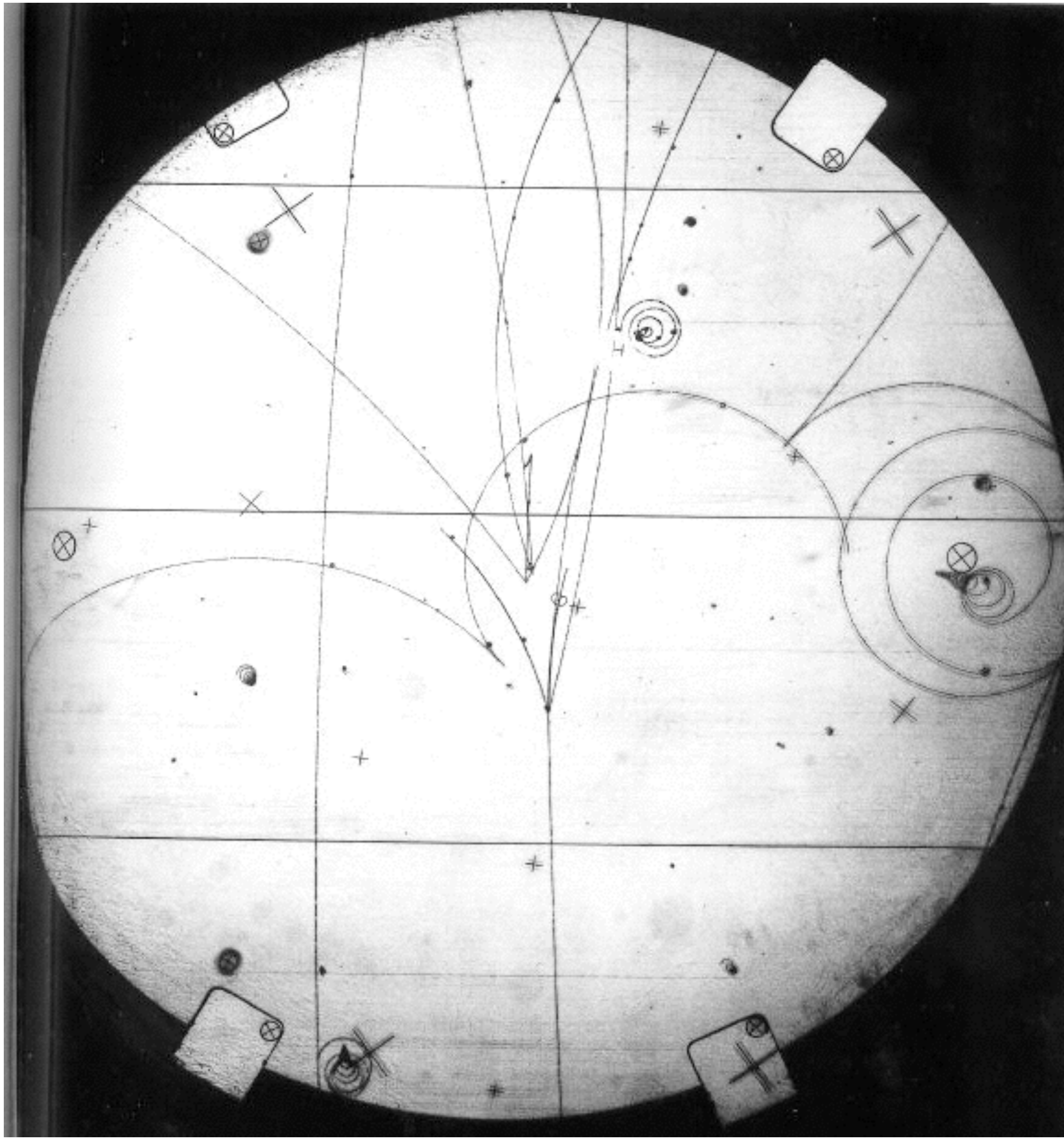
Dirac originally proposed that this might be the proton

Excuse to Triumph

- 1932: Anderson finds “positrons” in cosmic rays
- Exactly like electrons but positively charged: Fits what Dirac was looking for



Dirac predicts the existence of anti-matter and it is found



Solutions to the Dirac Equation at Rest:

$$\psi_1(t) = e^{-imc^2t/\hbar} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

"spin up"

$$\psi_2(t) = e^{-imc^2t/\hbar} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

"spin down"

positive energy solutions (particle)

$$\psi_3(t) = e^{+imc^2t/\hbar} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

"spin down"

$$\psi_4(t) = e^{+imc^2t/\hbar} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

"spin up"

"negative" energy solutions (anti-particle)

- Note that all particles have the same mass

Pedagogical Sore Point

- All the discussion we had thus far about difficulties with relativistic equations, (negative probabilities, negative energies) is of historic interest
- Scientifically, the framework for dealing with quantum mechanics and special relativity (i.e. quantum field theory) had not been developed
  - The old tools of NR quantum mechanics had reached their limit and new ones were necessary.
  - In particular, the idea of a “wavefunction” had to be revisited
  - Until this was done, there were many difficulties!
  - Once QFT was developed, all of these problems go away.
- Both KG and Dirac Equations are valid in QFT
  - No negative probabilities, no negative energies
- Nonetheless, the history and its course are rather interesting.

Plane Wave Solutions to the Dirac Equation:

- Consider a solution of the form:

column vector

$$\psi(x) = e^{-ik\cdot x}u(k)$$

Column vector of 4 elements with space-time dependence

space-time dependence

- and place it in the Dirac equation:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0$$

$$(\gamma^\mu\hbar k_\mu - mc)e^{-ik\cdot x}u(k) = 0$$

$$(\gamma^\mu\hbar k_\mu - mc)u(k) = 0$$

What does this equation look like:

$$\gamma^\mu k_\mu = \gamma^0 k^0 - \gamma^1 k^1 - \gamma^2 k^2 - \gamma^3 k^3$$

- we found this is:

$$\begin{pmatrix} k_0 & -\mathbf{k}\cdot\sigma \\ \mathbf{k}\cdot\sigma & -k_0 \end{pmatrix}$$

Note 2x2 notation

$$u(k) \Rightarrow \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

- So that the Dirac equation reads:

$$\begin{pmatrix} \hbar k_0 - mc & -\hbar\mathbf{k}\cdot\sigma \\ \hbar\mathbf{k}\cdot\sigma & -\hbar k_0 - mc \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$(\hbar k_0 - mc)u_A - \hbar\mathbf{k}\cdot\sigma u_B = 0$$

$$u_A = \frac{\hbar\mathbf{k}\cdot\sigma}{(\hbar k_0 - mc)} u_B$$

$$\hbar\mathbf{k}\cdot\sigma u_A - (\hbar k_0 + mc)u_B = 0$$

$$\frac{\hbar\mathbf{k}\cdot\sigma}{(\hbar k_0 + mc)} u_A = u_B$$

Determing u

$$u_A = \frac{\hbar\mathbf{k}\cdot\sigma}{(\hbar k_0 - mc)}u_B \qquad \frac{\hbar\mathbf{k}\cdot\sigma}{(\hbar k_0 + mc)}u_A = u_B \qquad \frac{\hbar\mathbf{k}\cdot\sigma}{(\hbar k_0 + mc)}\frac{\hbar\mathbf{k}\cdot\sigma}{(\hbar k_0 - mc)} = 1$$

- this means we can identify ħk ↔ p

$$\begin{aligned}\hbar^2\mathbf{k}^2 &= (\hbar k_0)^2 - m^2c^2\\ \mathbf{p}^2 &= (E/c)^2 - m^2c^2\end{aligned} \qquad (p^0,\mathbf{p}) \Rightarrow (\pm\hbar k_0, \hbar\mathbf{k}) \qquad \mathbf{p}\cdot\sigma = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Use positive solutions

Use negative solutions

- We can now construct the column vector u:

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ p_z c/(E+mc^2) \\ (p_x+ip_y)c/(E+mc^2) \end{pmatrix} \qquad u_2 = N\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)c/(E+mc^2) \\ -p_z c/(E+mc^2) \end{pmatrix}$$

electrons

$$u_3 = N\begin{pmatrix} p_z c/(E-mc^2) \\ (p_x+ip_y)c/(E-mc^2) \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N\begin{pmatrix} (p_x-ip_y)c/(E-mc^2) \\ -p_z c/(E-mc^2) \\ 0 \\ 1 \end{pmatrix}$$

positrons

Normalization of the Wavefunction:

- We need to choose a standard "normalization" of the wavefunctions
  - Note that multiples of the solutions are still solution
  - The normalization convention simply fixes this arbitrary choice:

$$u^\dagger u = 2E/c \qquad u^\dagger \equiv (u^T)^* \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad \Rightarrow \quad u^\dagger = (u_1^*,\ u_2^*,\ u_3^*\ u_4^*)$$

  - for u_1

$$u_1^\dagger u_1 = N^2 \left[1 + \frac{\mathbf{p}^2}{(E+mc^2)^2}\right] = 2E/c \qquad N = \sqrt{(E+mc^2)/c}$$

Lorentz Properties:

- The Dirac equation "works" in all reference frames.
  - What exactly does this mean? "Lorentz Covariant"

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0$$

- i, $\hbar$, m and c are constants that don't change with reference frames.
- ∂_μ and ψ will change with reference frames, however.
  - ∂_μ is a derivative that will be taken with respect to the space-time coordinates in the new reference frame. We'll call this ∂'_μ
  - how does ψ change?
    - $\psi' = S\psi$ where ψ' is the spinor in the new reference frame
- Putting this together, we have the following transformation of the equation when evaluating it in a new reference frame

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \qquad \Rightarrow \qquad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

What properties does S need to make this work?

The Properties of S

- Since we know how to relate space time coordinates in one reference with another (i.e. Lorentz transformation), we can do the same for the derivatives
- Using the chain rule, we get: $\partial'_\mu \equiv \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}} \frac{\partial}{\partial x^\nu}$
  - where we view x as a function x' (i.e. the original coordinates as a function of the transformed or primed coordinates).
  - Note the summation over ν
- if the primed coordinates moving along the x axis with velocity βc:

$$\begin{aligned} x^0 &= \gamma(x^{0\prime} + \beta x^{1\prime}) \\ x^1 &= \gamma(x^{1\prime} + \beta x^{0\prime}) \\ x^2 &= x^{2\prime} \\ x^3 &= x^{3\prime} \end{aligned}$$

$$(\nu = 0, \mu = 0) \Rightarrow \frac{\partial x^0}{\partial x^{0\prime}} = \gamma$$

$$(\nu = 0, \mu = 1) \Rightarrow \frac{\partial x^0}{\partial x^{1\prime}} = \gamma\beta$$

etc.

Transforming the Dirac Equation:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0 \qquad \Rightarrow \qquad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\,\psi' = 0$$

$$i\hbar\gamma^\mu\partial'_\mu(S\psi) - mc\,(S\psi) = 0$$

$$i\hbar\gamma^\mu\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu(S\psi) - mc\,(S\psi) = 0$$

S is constant in space time, so we can move it to the left of the derivatives

$$i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,(S\psi) = 0$$

Now slap S^{-1} from both sides

$$i\hbar\gamma^\nu\partial_\nu\psi - mc\,\psi = 0 \qquad S^{-1} \rightarrow i\hbar\gamma^\mu S\frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu\psi - mc\,S\psi = 0$$

Since these equations must be the same, S must satisfy

$$\gamma^\nu = S^{-1}\gamma^\mu S\ \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

Example: The parity operator

- For the parity operator, we want to invert the spatial coordinates while keeping the time coordinate unchanged:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\frac{\partial x_0}{\partial {x_0}'} = 1 \qquad \frac{\partial x_1}{\partial {x_1}'} = -1$$

$$\frac{\partial x_2}{\partial {x_2}'} = -1 \qquad \frac{\partial x_3}{\partial {x_3}'} = -1$$

- We then have

$$\gamma^0 = S^{-1}\gamma^0 S$$
$$\gamma^1 = -S^{-1}\gamma^1 S$$
$$\gamma^2 = -S^{-1}\gamma^2 S$$
$$\gamma^3 = -S^{-1}\gamma^3 S$$

Recalling

$$\gamma^\nu = S^{-1}\gamma^\mu S\,\frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\gamma^0)^2 = 1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0$$

We find that γ^0 satisfies our needs

$$\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

$$\gamma^i = -\gamma^0\gamma^i\gamma^0 = \gamma^0\gamma^0\gamma^i = \gamma^i$$

$$S_P = \gamma^0$$

Next time

- Read 7.1-7.4
- I would encourage you to work out the examples in 7.6 yourself explicitly so that you start to gain some fluency with the Feynman rules
- Lots of notation, lots of stuff going on . . . .
    - please stop by if you have questions!