Lecture 11: ABC Theory and Feynman Diagrams

H. A. Tanaka

Reminder

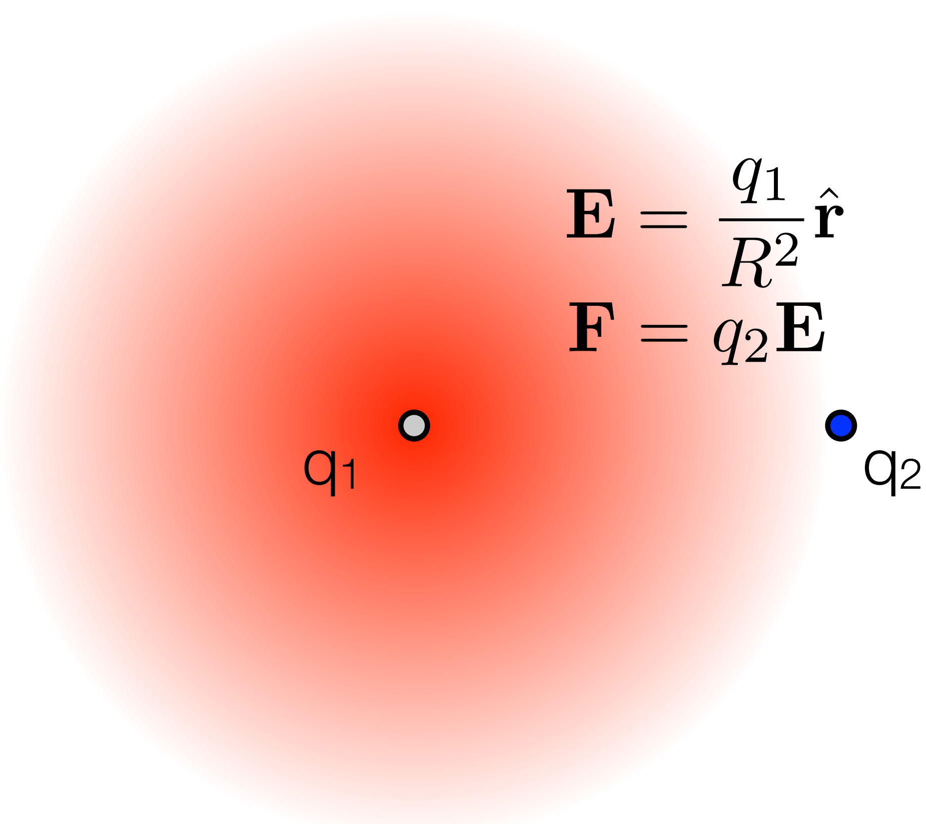
- Problem Set 2 due in Box 7 (basement) today by 1700.

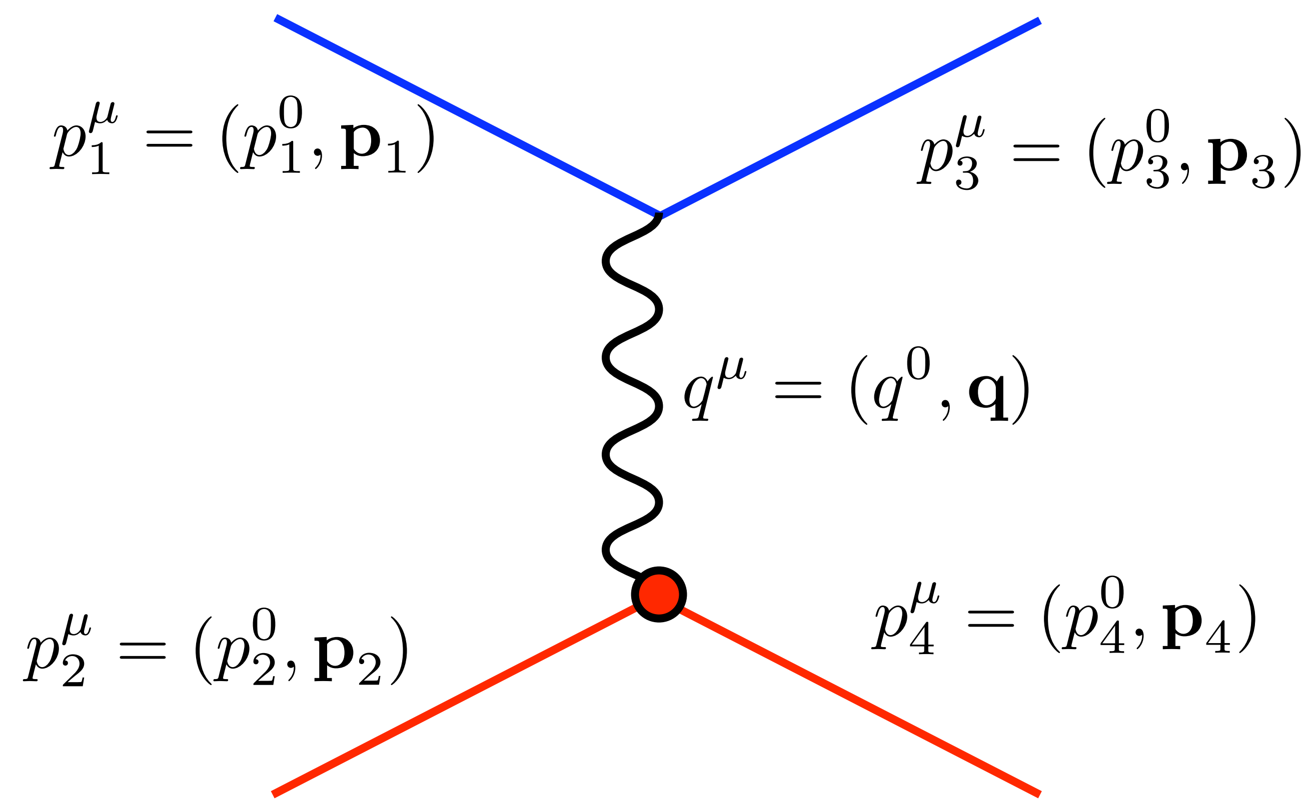
Today:

- Introduction into Feynman rules for a basic theory
  - We’ve already played around with Feynman diagrams
  - Start of the process to learn how to turn these into definite expressions for the amplitude of a process
  - This has been the missing ingredient in our study so far.
    - this will be the bulk of the rest of the class
- Introduce the “ABC” theory
  - Realistic theory three species of scalar (spin 0) particles (A, B, C) with an interaction
  - but it doesn’t correspond to any interaction/particles that we know of
  - we study it because it is particularly simple and allows us to study the “recipe” without too much mathematical complication.
  - We’ll see a “real” theory (QED) in the next chapter.

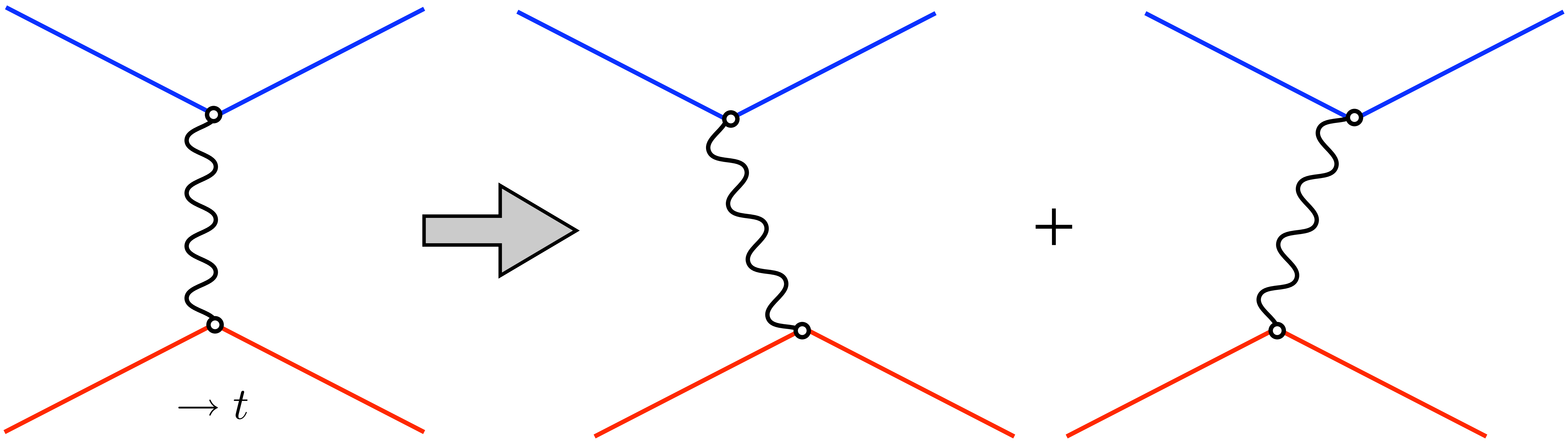
Basic Review of Particle Dynamics

- Interactions between particles are effected by the exchange of particles
  - Electromagnetic interactions are the result of the exchange of photons between electrically charged particles:





Time ordering of vertices



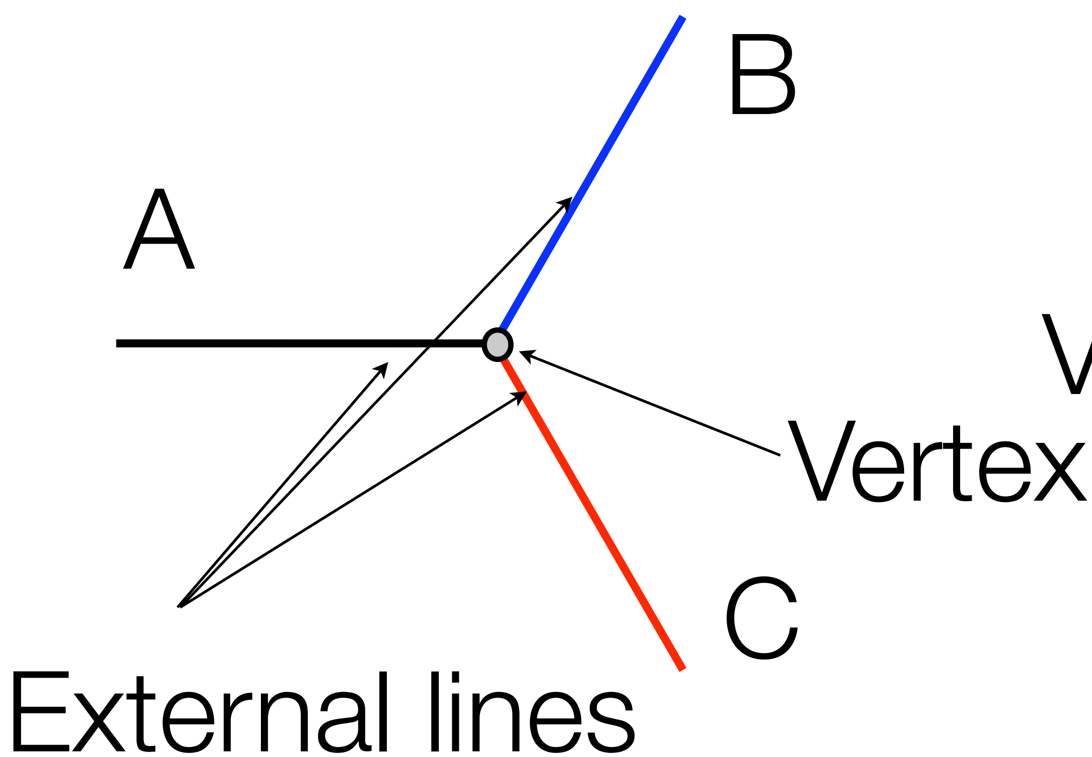
- Feynman diagrams incorporate possible time orderings
    - The vertical exchange illustrates that the amplitude corresponding to the diagram is agnostic as to which “direction the exchange particle goes”
    - The derivation of the Feynman rules through Quantum Field Theory includes this into consideration.

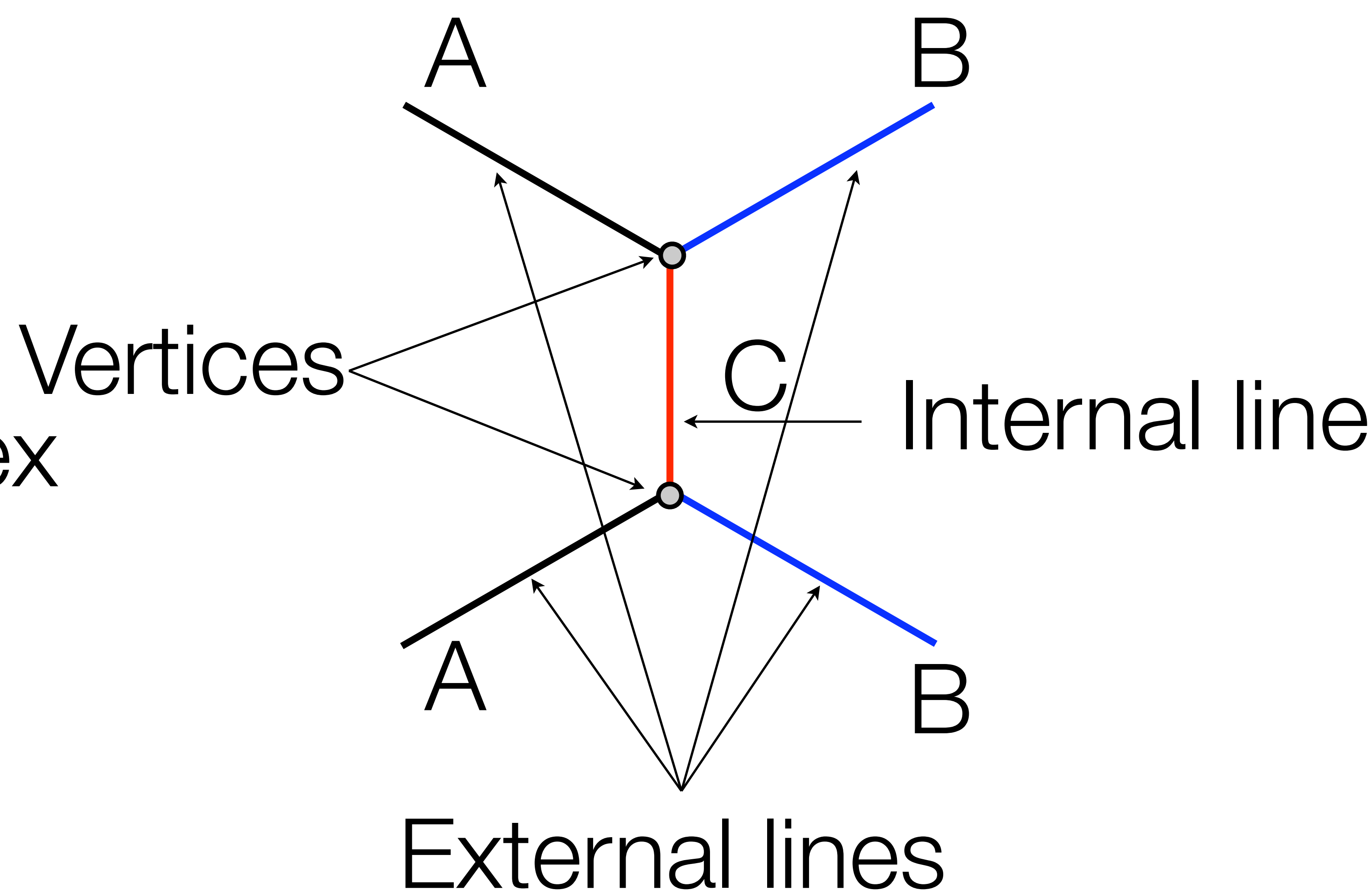
Components of a Feynman Diagram

- **External Lines:**
  - particles that come in and out in the initial and final state, respectively.
  - for spin-0, there is no factor.
- **Vertex factors:**
  - each vertex (i.e. where A, B, C meet) has a factor.
  - determines "order" of diagram: order=number of vertices
- **Internal lines and Propagators:**
  - Factors for internal particles exchanged between vertices
  - Only applies to particles "internal" to the diagram, not external lines
- **Momentum Conservation at vertex:**
  - (4d) delta function at each vertex enforcing 4-momentum conservation
  - Integrals over internal momentum:
- Internal lines have any momentum consistent with 4-momentum conservation

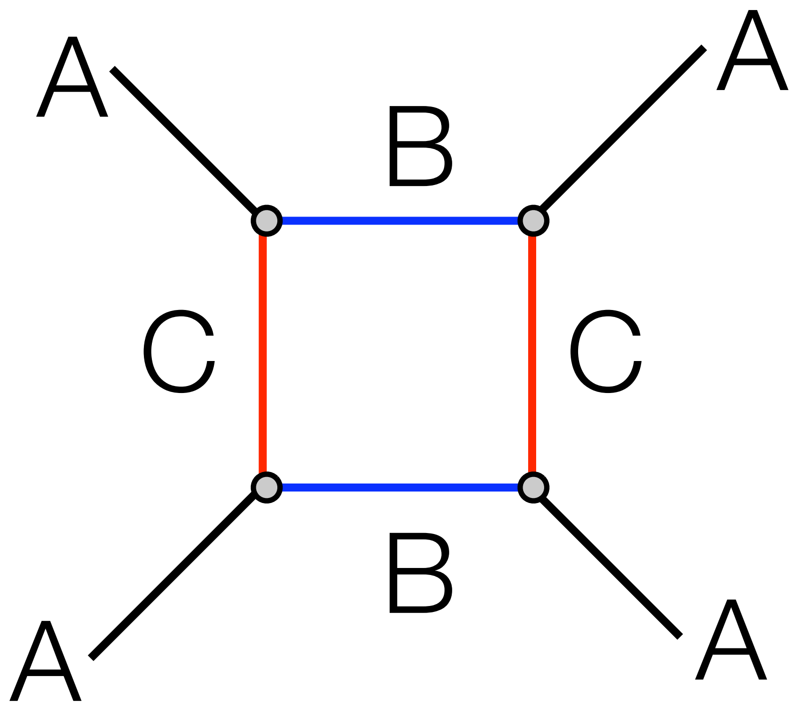
Basic Structure

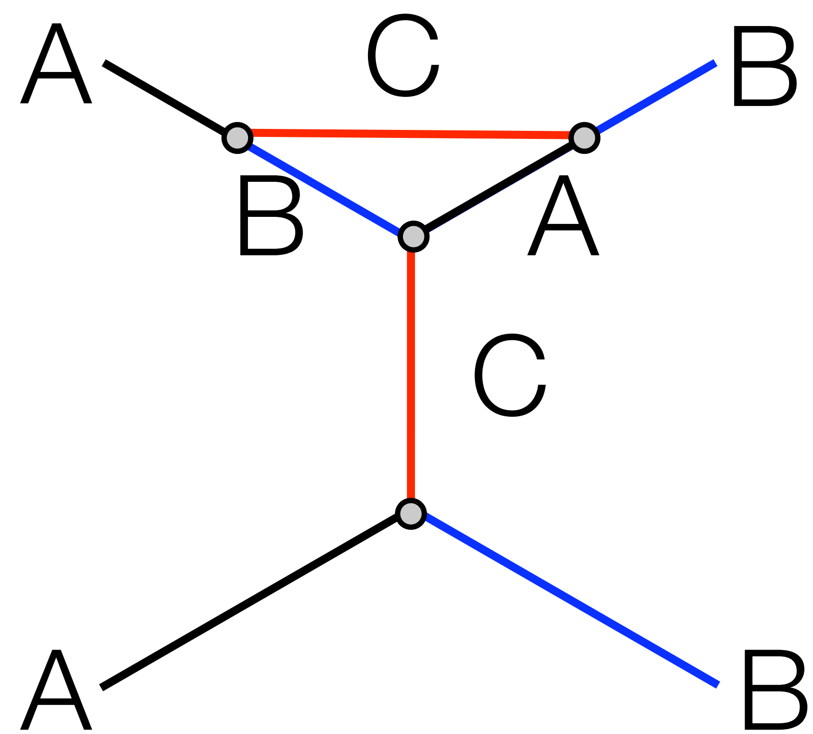
Theory of 3 “scalar” (spin 0) particles that are distinct with one interaction
A, B, C are different types of particles

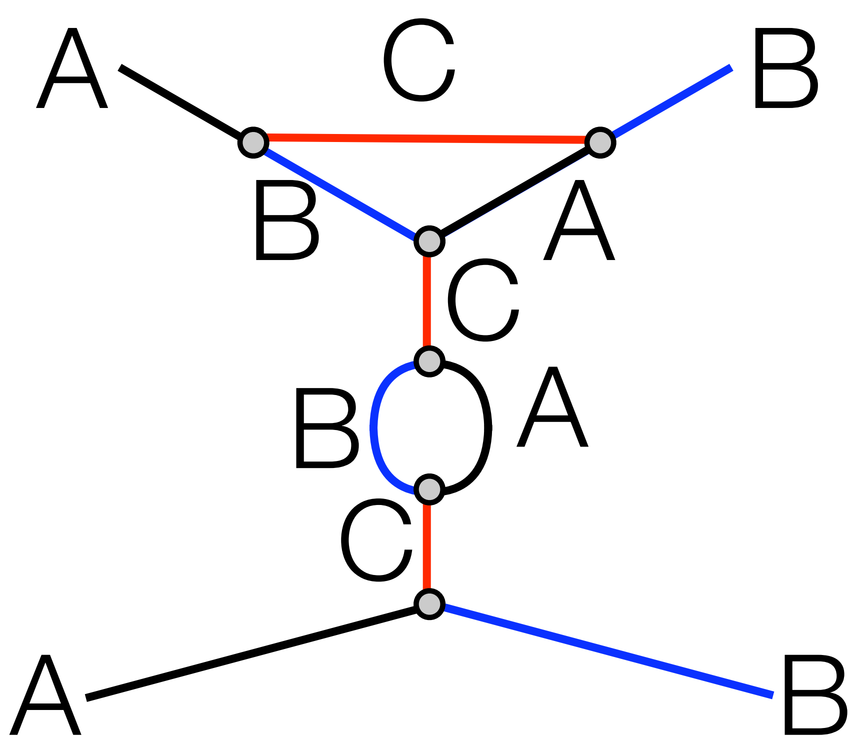




More examples

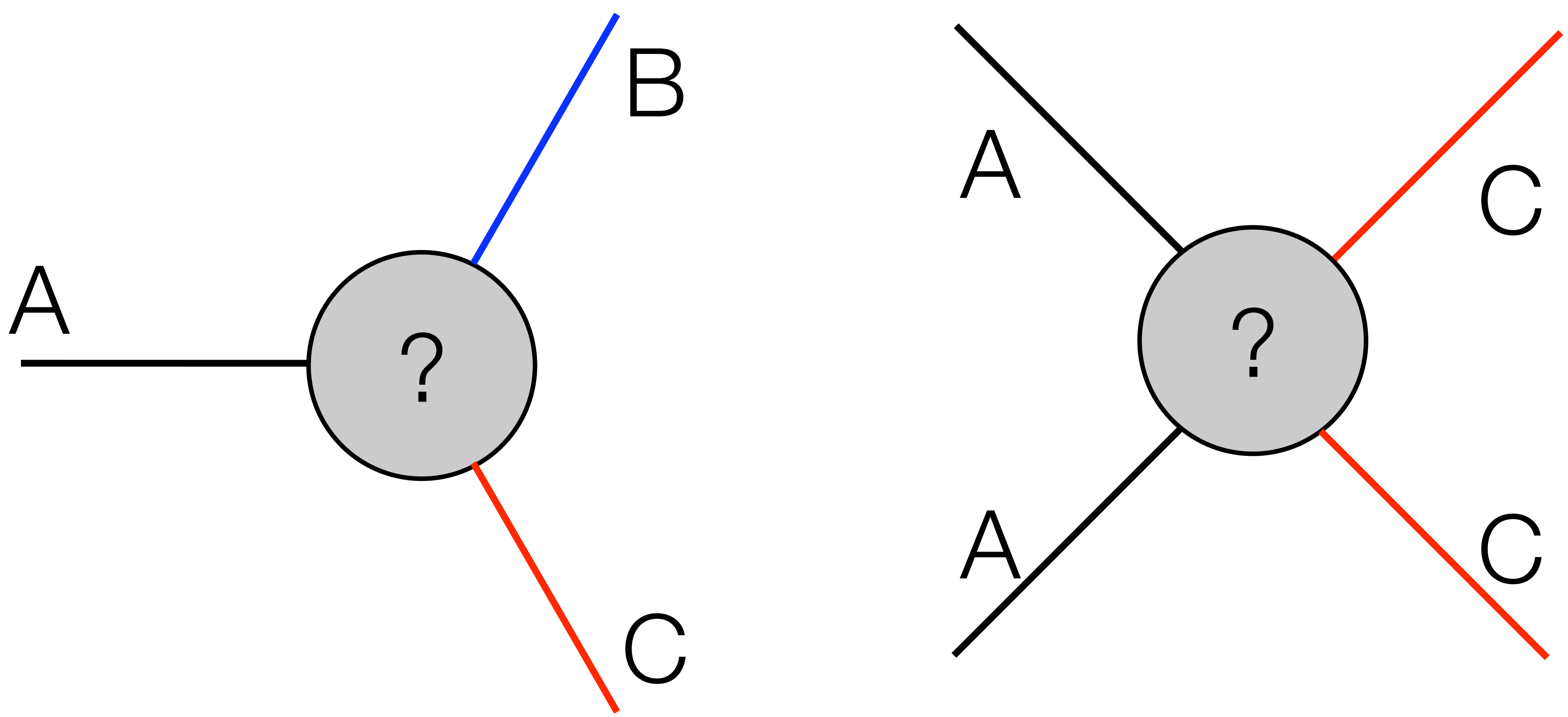






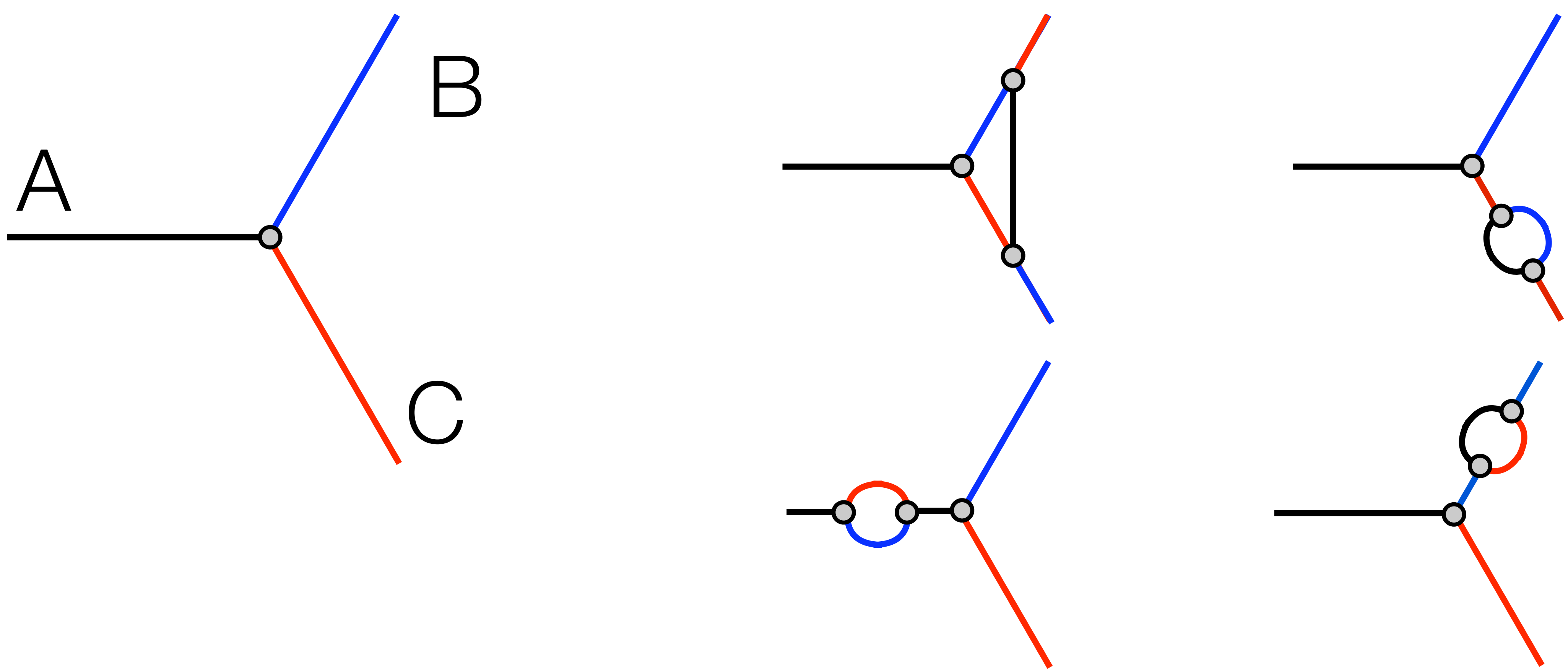
Step 1

- For a given process (e.g. A→B+C, A+C→A+C), specify the external lines:



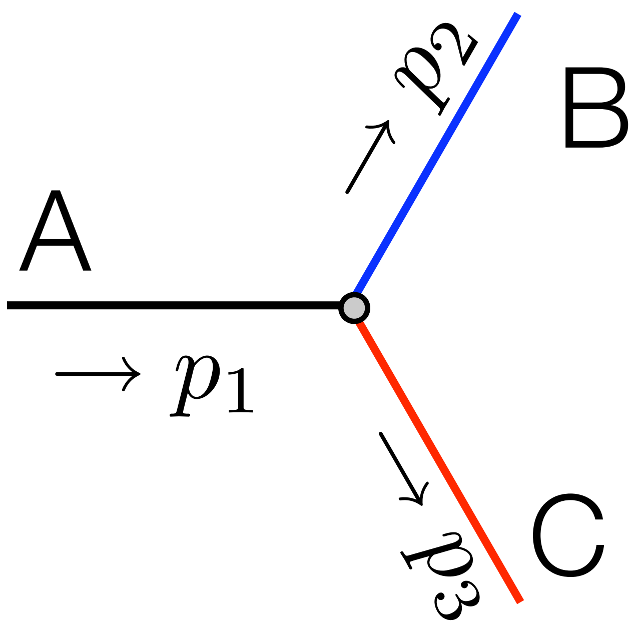
Step 2: diagram by “order”

- Identify all diagrams that contain ≤N vertices, where N is the order we want to calculate the amplitude to:

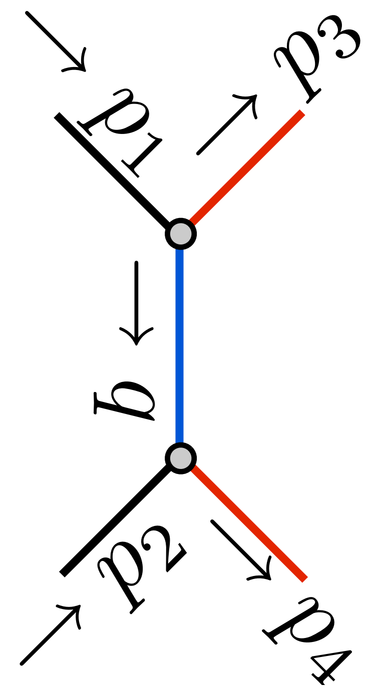


- Check that each vertex follows the vertex “rules”
- There are no second order diagrams for this process

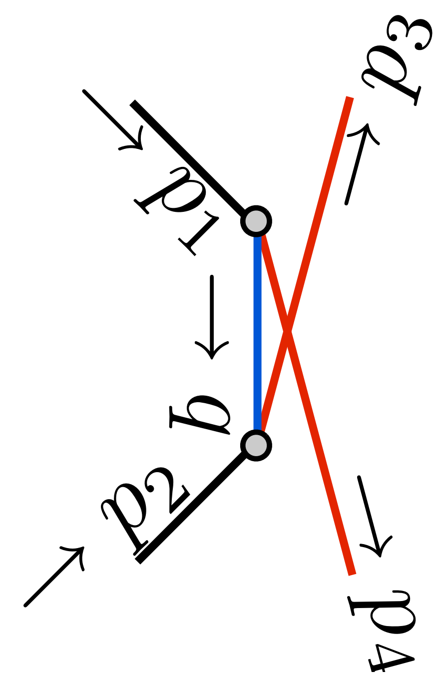
Step 3: Label the momentum flow, vertex factors



$-ig$



$(-ig)^2$

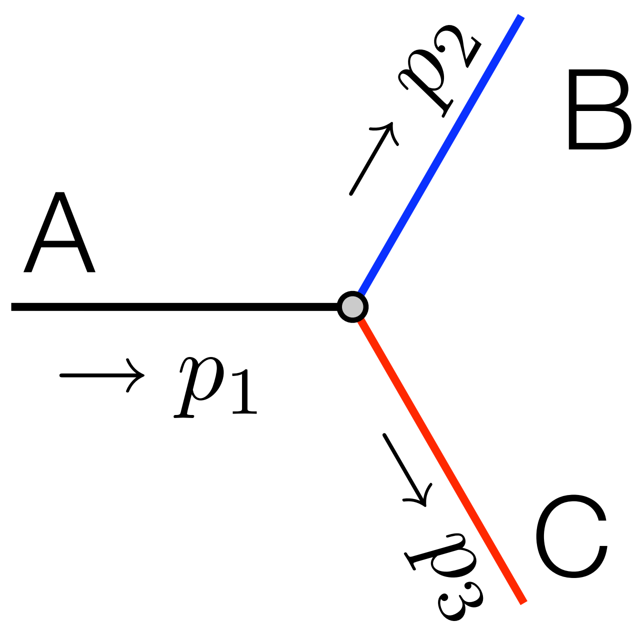


$(-ig)^2$

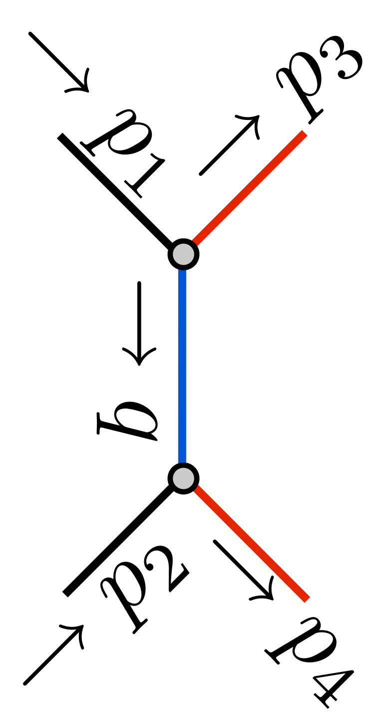
- We can direct the arrows in any way so long as we are consistent
- **Introduce a factor of (-ig) for each vertex. (Feynman rule for vertex)**

Step 4: Propagators

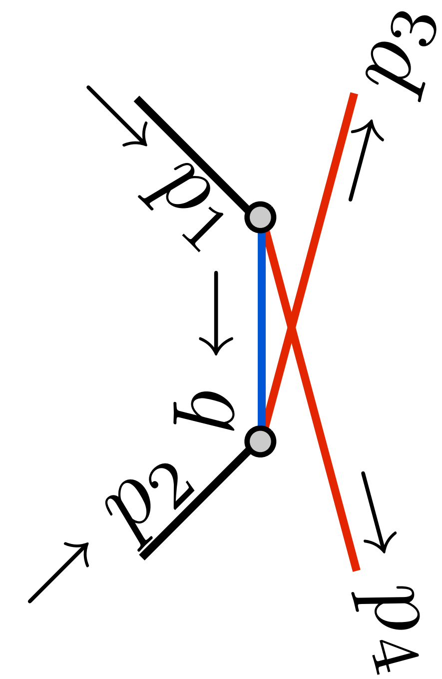
- Introduce a propagator term for each internal line



$$(-ig)$$



$$(-ig)^2 \frac{i}{q^2 - m_B^2 c^2}$$

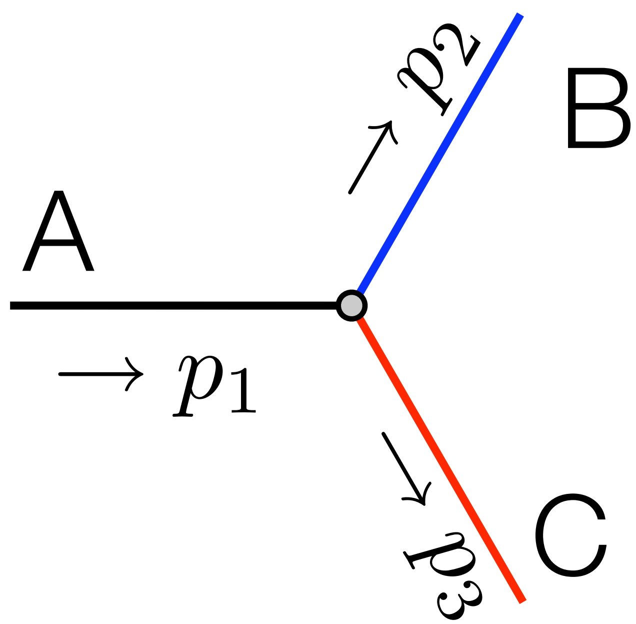


$$(-ig)^2 \frac{i}{q^2 - m_B^2 c^2}$$

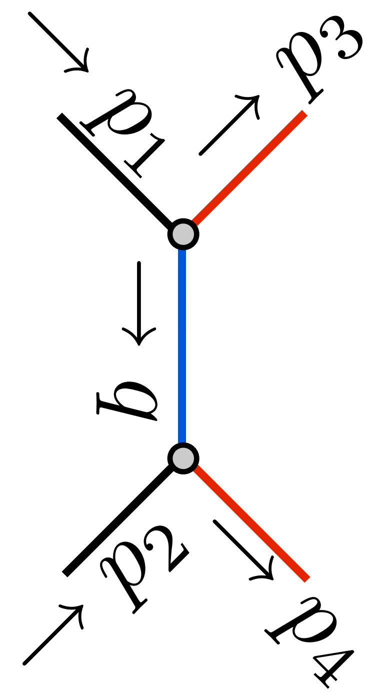
- Note that we have to use the momentum we assigned to the line!

Step 5: energy-momentum conservation

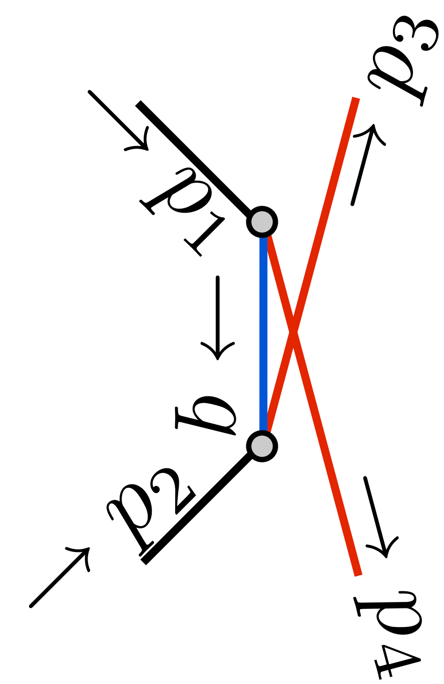
- Introduce a delta function enforcing 4-momentum conservation at each vertex



$$(-ig) \times (2\pi)^4 \; \delta^4(p_1 - p_2 - p_3)$$



$$(-ig)^2 \frac{i}{q^2 - m_B^2 c^2}$$
$$\times \, (2\pi)^4 \delta^4(p_1 - p_3 - q)$$
$$\times \, (2\pi)^4 \delta^4(p_2 - p_4 + q)$$

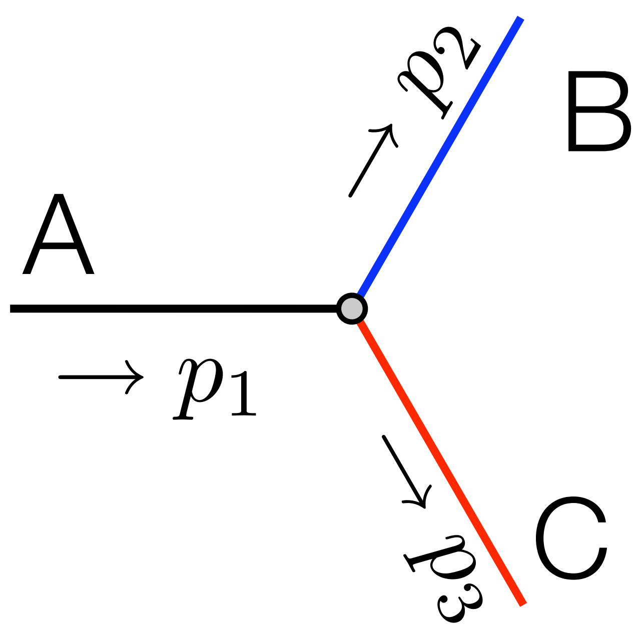


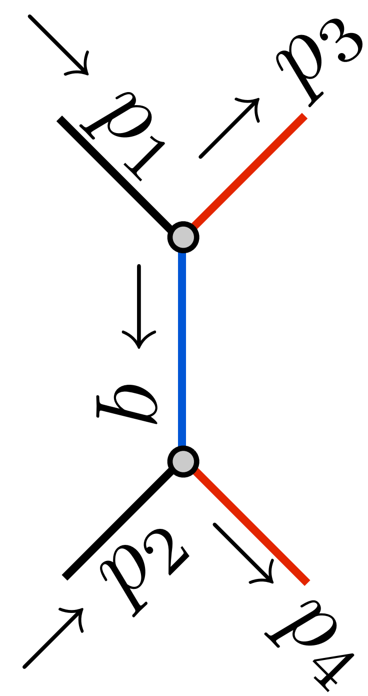
$$(-ig)^2 \frac{i}{q^2 - m_B^2 c^2}$$
$$\times \, (2\pi)^4 \delta^4(p_1 - p_4 - q)$$
$$\times \, (2\pi)^4 \delta^4(p_2 - p_3 + q)$$

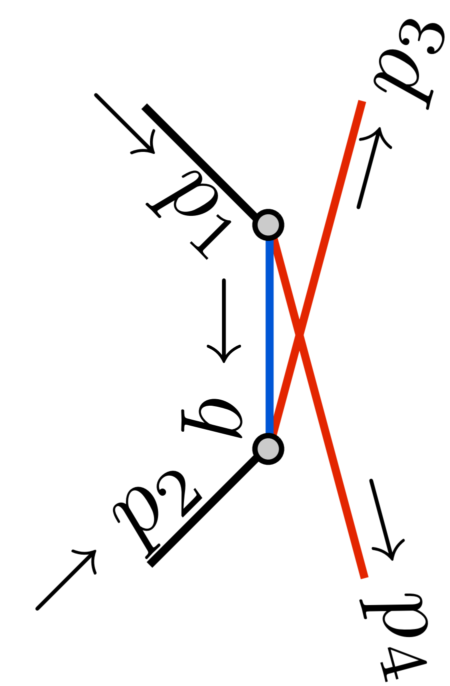
- convention: momentum in/out of vertex is (+/-)

Step 6: integrate over internal momenta:

- Introduce an integral over the momentum of each internal line:







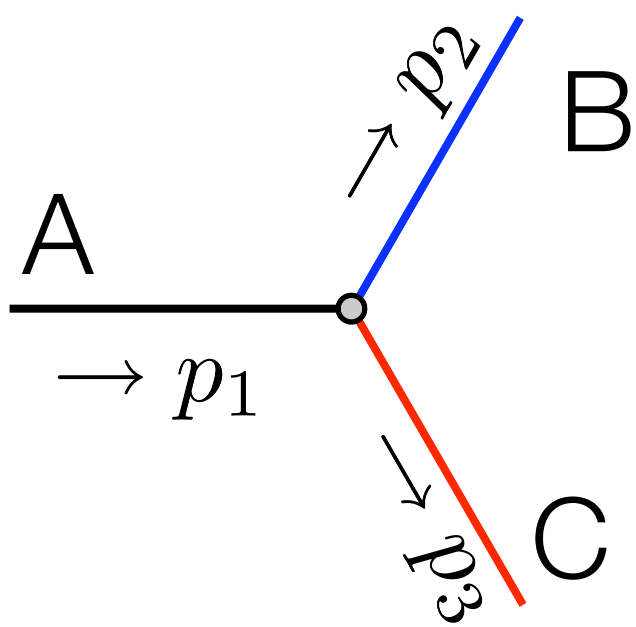
$$(-ig) \times (2\pi)^4 \; \delta^4(p_1 - p_2 - p_3)$$

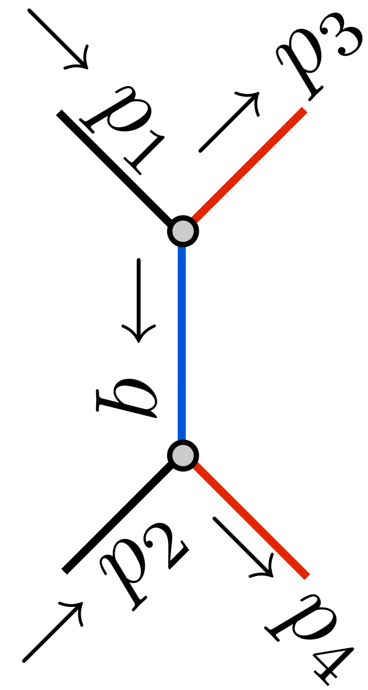
$$\int \frac{1}{(2\pi)^4} d^4q (-ig)^2 \frac{i}{q^2 - m_B^2 c^2} \times (2\pi)^4 \delta^4(p_1 - p_3 - q) \times (2\pi)^4 \delta^4(p_2 - p_4 + q)$$

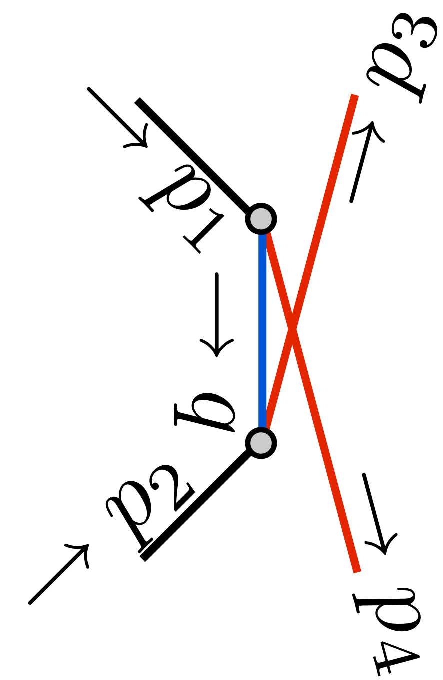
$$\int \frac{1}{(2\pi)^4} d^4q (-ig)^2 \frac{i}{q^2 - m_B^2 c^2} \times (2\pi)^4 \delta^4(p_1 - p_4 - q) \times (2\pi)^4 \delta^4(p_2 - p_3 + q)$$

Step 7: perform the integral

- Perform the integrals, leaving a δ function for overall 4-p conservation







$$(-ig) \times (2\pi)^4\ \delta^4(p_1 - p_2 - p_3)$$

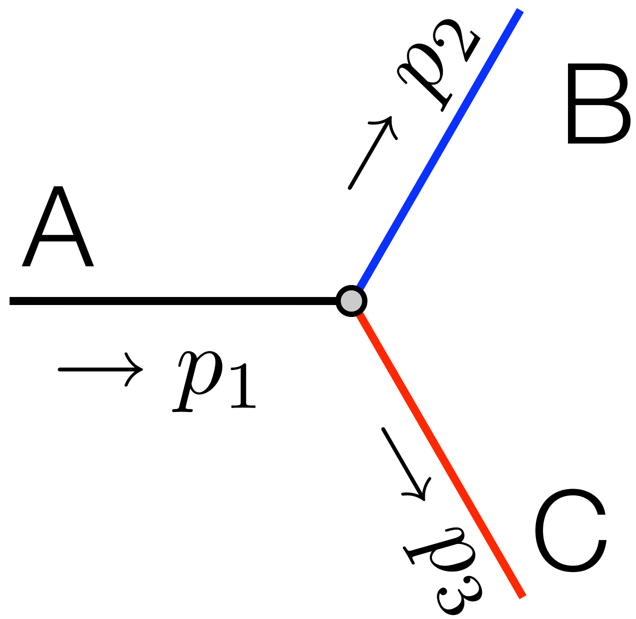
$$(-ig)^2 \frac{i}{(p_4 - p_2)^2 - m_B^2 c^2} \times (2\pi)^4 \delta^4(p_1 - p_3 - p_4 + p_2)$$

$$(-ig)^2 \frac{i}{(p_3 - p_2)^2 - m_B^2 c^2} \times (2\pi)^4 \delta^4(p_1 - p_3 - p_4 + p_2)$$

- In lowest order calculations, each integral will meet with a delta function.

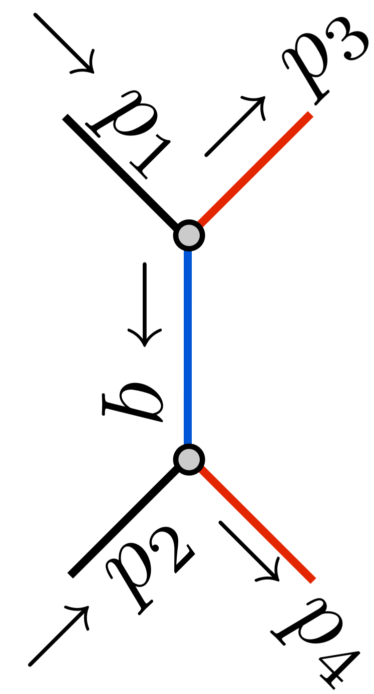
Result:

- Eliminate the δ function, and the Resulting expression is -iM



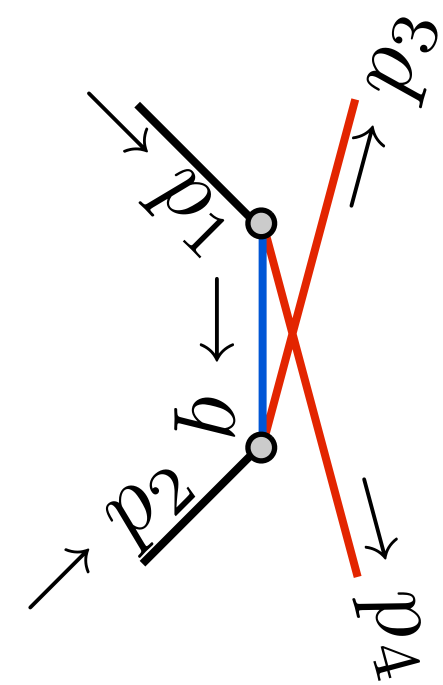
$$(-ig)$$

$$\mathcal{M} = g$$



$$(-ig)^2 \frac{i}{(p_4 - p_2)^2 - m_B^2 c^2}$$

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_B^2 c^2}$$



$$(-ig)^2 \frac{i}{(p_3 - p_2)^2 - m_B^2 c^2}$$

$$\mathcal{M} = \frac{g^2}{(p_3 - p_2)^2 - m_B^2 c^2}$$

Decay Rate of A→B+C

- Recall our formula for the two-body decay of a particle:

$$\Gamma = \frac{S}{8\pi\hbar m_1^2 c} \times |\mathcal{M}|^2 \times \ |\mathbf{p_2}|$$

- Now that we can calculate the amplitude, we can finish our calculation:

$$\Gamma = \frac{1}{8\pi\hbar m_A^2 c} \times g^2 \times |\mathbf{p}_{B/C}|$$

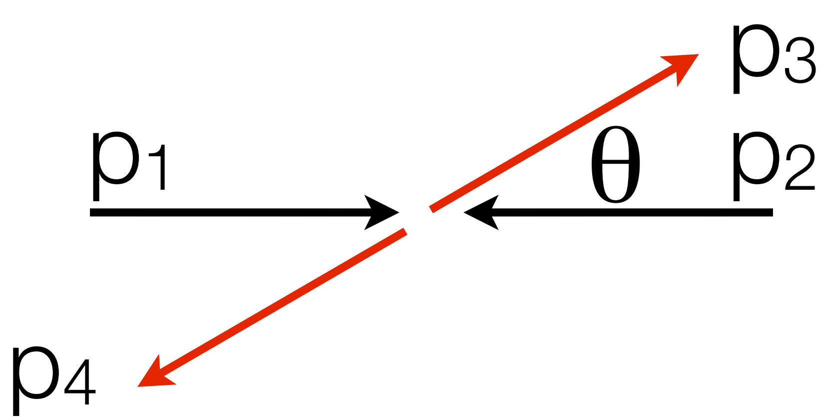
- from PS 2

$$|\mathbf{p}_{B/C}| = \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

Scattering rate of A+A→C+C in CM Frame

- From the textbook (6.47), assume $M=M_A=M_C$, $M_B = 0$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$



$$\mathcal{M} = g^2\left[\frac{1}{(p_4-p_2)^2 - m_B^2c^2} + \frac{1}{(p_3-p_2)^2 - m_B^2c^2}\right]$$

$$(p_4-p_2)^2 = m_C^2c^2 + m_A^2c^2 - 2p_4\cdot p_2 = m_C^2c^2 + m_A^2c^2 - 2((E/c)^2 - \mathbf{p}_4\cdot\mathbf{p}_2)$$
$$= m_C^2c^2 + m_A^2c^2 - 2((E/c)^2 - |\mathbf{p}_f|\cdot|\mathbf{p}_i|\cos\theta)$$

$$(p_3-p_2)^2 = m_C^2c^2 + m_A^2c^2 - 2p_3\cdot p_2 = m_C^2c^2 + m_A^2c^2 - 2((E/c)^2 - \mathbf{p}_3\cdot\mathbf{p}_2)$$
$$= m_C^2c^2 + m_A^2c^2 - 2((E/c)^2 + |\mathbf{p}_f|\cdot|\mathbf{p}_i|\cos\theta)$$

$$(p_4-p_2)^2 = 2m^2c^2 - 2((E/c)^2 - |\mathbf{p}|^2\cos\theta) = -2|\mathbf{p}|^2(1-\cos\theta)$$
$$(p_3-p_2)^2 = 2m^2c^2 - 2((E/c)^2 + |\mathbf{p}|^2\cos\theta) = -2|\mathbf{p}^2|(1+\cos\theta)$$

Endgame

$$\mathcal{M} = g^2 \left[\frac{1}{-2|\mathbf{p}|^2(1-\cos\theta)} + \frac{1}{-2|\mathbf{p}|^2(1+\cos\theta)} \right]$$

$$= g^2 \frac{-2|\mathbf{p}|^2(1+\cos\theta) - 2|\mathbf{p}|^2(1-\cos\theta)}{4|\mathbf{p}|^4(1-\cos^2\theta)}$$

$$= \frac{-g^2}{|\mathbf{p}|^2 \sin^2\theta}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \qquad S = 1/2,\ E_1 = E_2,\ |\mathbf{p}_f| = |\mathbf{p}_i|$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{\hbar c}{8\pi}\right)^2 \frac{g^4}{(2E^2|\mathbf{p}|^2\sin^2\theta)^2}$$