Decay, Scattering and Phase Space II

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Announcements

- Problem set 2 due on Tuesday 1700 in Drop Box.

Final Result: Total two-body decay rate:

typo in slides

$$\Gamma = \frac{S}{8\pi\hbar m_1^2 c} \times |\mathcal{M}|^2 \times |\vec{p}_2|$$

- We now need to be able to calculate the matrix element M
  - Thus far (in isospin, etc.) we have dealt with relationships between different M based on symmetry rather than calculating M directly
  - later, we'll use Feynman diagrams and the associated calculus to calculate amplitudes for various elementary processes

Two-Body Decay:

- Last time, we derived the rate for two-body decay

$$\Gamma = \frac{S}{8\pi\hbar m_1^2 c} \times |\mathcal{M}|^2 \times |\vec{p}_2|$$

- Let's put this equation to use to consider the strong decay of a particle
  - An example of this would be $\rho \to \pi^+\pi^-$
  - We have all the quantities (p_2 needs to be consistent with E conservation)
    - except the amplitude
  - In the past, people had to "guess" the amplitude.
  - Looking at the units:

$$\frac{1}{T} \sim \frac{|\mathcal{M}|^2 \times P}{E \times T \times M^2 \times c} \Rightarrow |\mathcal{M}|^2 \sim \frac{E \times M^2 \times c}{P} \Rightarrow E \times m \Rightarrow P^2$$

    - The only momentum in the problem is $|p_2|$.
    - We could also consider $M_\rho,\ v_\pi$
  - The coupling is strong so let's assume $\alpha_s \sim 1$

Calculating the Lifetime

- Calculate p_2 $E_2 = \frac{1}{2} M_\rho c^2 \quad \Rightarrow \quad p_2 = c\sqrt{(\frac{1}{2} M_\rho)^2 - m_\pi^2}$

$$\Gamma = \frac{S}{8\pi\hbar m_1^2 c} \times |\mathcal{M}|^2 \times \; p_2 = \frac{(385^2 \text{ MeV}^2/c^2 - 139^2 \text{ MeV}^2/c^2)^{3/2}}{8\pi \times 6.58 \times 10^{-20} \text{MeV sec } (770 \text{ MeV}^2/c^4)c}$$

$$\frac{10^5 \text{MeV}^3/c^3}{1.27 \times 10^5 \times 10^{-20} \text{MeV}^3/c^3 \text{sec}}$$

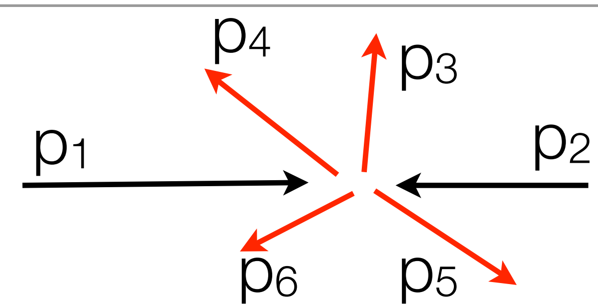
$$\Gamma \sim 10^{20}/\text{sec}$$

- This sets the typical time scale for strong decays unless the phase space is dramatically different.

- To do better, we need a better idea of the amplitude/matrix element.

Scattering:

p$_1$ and p$_2$ are 4-vectors!



$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \, \delta^4(p_1^\mu + p_2^\mu - \sum_{f=3} p_f^\mu)$$

$$\times \prod_{f=3}^{N} 2\pi \, \delta(p_f^2 - m_f^2 c^2) \, \Theta(p_f^0) \, \frac{d^4 p_f}{(2\pi)^4}$$

- Look somewhat familiar?

$$\Gamma = \frac{S}{2\hbar m} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \, \delta^4(p_1^\mu - \sum_{f=2}^{N} p_f^\mu)$$

$$\times \prod_{f=2}^{N} 2\pi \, \delta(p_f^2 - m_f^2 c^2) \, \Theta(p_f^0) \, \frac{d^4 p_f}{(2\pi)^4}$$

- It has almost the same form as the decay phase space

Integrating to enforce mass/energy>0

- The integral over dp_f^0 turns out the exactly the same as before

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu) \times \prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2 c^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3}$$

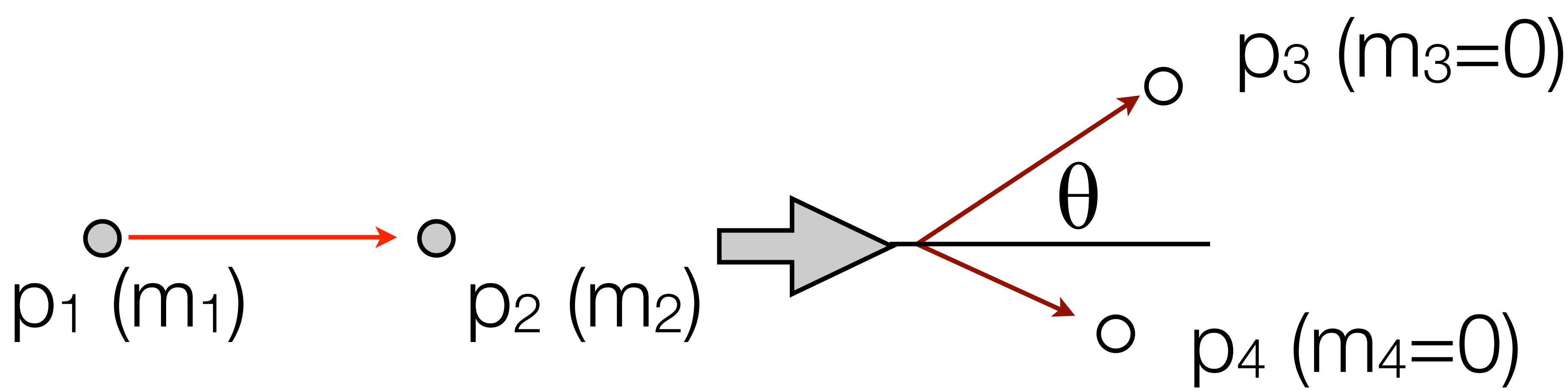
$$\Updownarrow$$

$$\Gamma = \frac{S}{2\hbar m_1} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \; \delta^4(p_1^\mu - \sum_{f=2}^{N} p_f^\mu) \prod_{f=2}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3}$$

Griffiths Problem 6.9

- Consider the collision 1 + 2→ 3+ 4 in the lab frame (2 at rest), with particles 3 and 4 massless. Obtain the differential cross section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2|\mathbf{p}_3|}{m_2|\mathbf{p}_1|(E_1 + m_2c^2 - \mathbf{p}_1 c\cos\theta)}$$



- start with the general scattering phase space formula

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1\cdot p_2)^2 - (m_1m_2c^2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4\delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu) \times \prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2c^2}}\frac{d^3\mathbf{p}_f}{(2\pi)^3}$$

Evaluate the kinematic term

$$\sigma = \frac{S\hbar^2}{\boxed{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu) \times \prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2 c^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3}$$

$$\begin{aligned} p_1 &= (E_1/c, \mathbf{p}_1) \\ p_2 &= (E_2/c, \mathbf{p}_2) = (m_2 c, \mathbf{0}) \end{aligned}$$

$$p_1 \cdot p_2 = E_1 m_2$$

$$(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2 = E_1^2 m_2^2 - m_1^2 m_2^2 c^4 = m_2^2 c^2 (E_1^2/c^2 - m_1^2 c^2)$$

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \sqrt{m_2^2 c^2 |\mathbf{p}_1|^2} = m_2 c |\mathbf{p}_1|$$

Work out the outgoing momentum differentials:

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu) \times \boxed{\prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2 c^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3}}$$

- We have two outgoing particles:

$$\prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2 c^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3} \to \frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} \frac{1}{2\sqrt{\mathbf{p}_4^2 + m_4^2 c^2}} \frac{d^3\mathbf{p}_3}{(2\pi)^3} \frac{d^3\mathbf{p}_4}{(2\pi)^3}$$

- and they are both massless

$$\to \frac{1}{2|\mathbf{p}_3|} \frac{1}{2|\mathbf{p}_4|} \frac{d^3\mathbf{p}_3}{(2\pi)^3} \frac{d^3\mathbf{p}_4}{(2\pi)^3}$$

What we have so far:

$$\sigma = \boxed{\frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}}} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \boxed{\delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu)} \times \boxed{\prod_{f=3}^{N} \frac{1}{2\sqrt{\mathbf{p}_f^2 + m_f^2 c^2}} \frac{d^3\mathbf{p}_f}{(2\pi)^3}}$$

$$m_2 c |\mathbf{p}_1|$$

$$\frac{1}{2|\mathbf{p}_3|} \frac{1}{2|\mathbf{p}_4|} \frac{d^3\mathbf{p}_3}{(2\pi)^3} \frac{d^3\mathbf{p}_4}{(2\pi)^3}$$

- Expand out the delta function

$$\delta^4(p_1^\mu + p_2^\mu - \sum_{f=3}^{N} p_f^\mu) \longrightarrow \delta^4(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)$$

$$\delta^0(p_1^0 + p_2^0 - p_3^0 - p_4^0) \times \delta^3(\mathbf{p}_1 + \mathbf{p}_2^\mu - \mathbf{p}_3^\mu - \mathbf{p}_4^\mu)$$

Carry out the integral over momentum

$$\sigma = \frac{S\hbar^2}{4m_2c|\mathbf{p_1}|} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0) \times \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \times \frac{1}{2|\mathbf{p}_3|} \frac{1}{2|\mathbf{p}_4|} \frac{d^3\mathbf{p}_3}{(2\pi)^3} \frac{d^3\mathbf{p}_4}{(2\pi)^3}$$

- Let's integrate over $\mathbf{p}_4$ since we want the answer in terms of $\mathbf{p}_3$
- The integral just sets $\mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{p}_1 - \mathbf{p}_3$ everywhere

$$\sigma = \frac{S\hbar^2}{4m_2c|\mathbf{p}_1|} \times \int |\mathcal{M}|^2 \times (2\pi)^4 \; \delta(E_1/c + m_2c - E_3/c - |\mathbf{p}_1 - \mathbf{p}_3|) \times \times \frac{1}{2|\mathbf{p}_3|} \frac{1}{2|\mathbf{p}_1 - \mathbf{p}_3|} \frac{d^3\mathbf{p}_3}{(2\pi)^6}$$

$$\sigma = \frac{S\hbar^2}{64\pi^2 m_2c|\mathbf{p}_1|} \times \int |\mathcal{M}|^2 \times \; \delta(E_1/c + m_2c - |\mathbf{p}_3| - |\mathbf{p}_1 - \mathbf{p}_3|) \times \frac{1}{|\mathbf{p}_3|} \frac{1}{|\mathbf{p}_1 - \mathbf{p}_3|} d^3\mathbf{p}_3$$

Dealing with p1 and p3

- Let's try to rexpress |**p**1-**p**3|

$$|\mathbf{p}_1 - \mathbf{p}_3| = \sqrt{(\mathbf{p}_1 - \mathbf{p}_3)\cdot(\mathbf{p}_1 - \mathbf{p}_3)}$$

$$|\mathbf{p}_1 - \mathbf{p}_3| = \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2\mathbf{p}_1\cdot\mathbf{p}_3} \quad = \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}$$

$$\sigma = \frac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|}\times\int|\mathcal{M}|^2\times\,\delta(E_1/c+m_2c-|\mathbf{p}_3|-|\mathbf{p}_1-\mathbf{p}_3|)\times\frac{1}{|\mathbf{p}_3|}\,\frac{1}{|\mathbf{p}_1-\mathbf{p}_3|}d^3\mathbf{p}_3$$

$$\sigma = \quad \tfrac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|}\times\int|\mathcal{M}|^2\times\;\delta(E_1/c+m_2c-|\mathbf{p}_3|-\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_3|^2-2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta})$$
$$\times\tfrac{1}{|\mathbf{p}_3|}\,\tfrac{1}{\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_3|^2-2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}d^3\mathbf{p}_3$$

$$\frac{d\sigma}{d\Omega} = \quad \tfrac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|}\times\int|\mathcal{M}|^2\times\;\delta(E_1/c+m_2c-|\mathbf{p}_3|-\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_3|^2-2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta})$$
$$\times|\mathbf{p}_3|\tfrac{1}{\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_3|^2-2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}d|\mathbf{p}_3|$$

Variable subsitution:

- Following the prescription from before:
  - examine the remaining delta function and set z to the energy of the outgoing particles:

$$\delta(E_1/c + m_2 c - |\mathbf{p}_3| - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta})$$

$$z = |\mathbf{p}_3| + \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}$$

$$\frac{dz}{d|\mathbf{p}_3|} = 1 + \frac{|\mathbf{p}_3| - |\mathbf{p}_1|\cos\theta}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}$$

$$\frac{dz}{d|\mathbf{p}_3|} = \frac{z - |\mathbf{p}_1|\cos\theta}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}$$

$$\frac{dz}{z - |\mathbf{p}_1|\cos\theta} = \frac{d|\mathbf{p}_3|}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}$$

Endgame

$$\frac{dz}{z - |\mathbf{p}_1|\cos\theta} = \boxed{\frac{d|\mathbf{p}_3|}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}}$$

$$z = \boxed{|\mathbf{p}_3| + \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}}$$

- Let's substitute

$$\frac{d\sigma}{d\Omega} = \frac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|} \times \int |\mathcal{M}|^2 \times \ \delta(E_1/c + m_2 c \boxed{- |\mathbf{p}_3| - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta})}$$
$$\times |\mathbf{p}_3| \boxed{\frac{1}{\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta}} d|\mathbf{p}_3|}$$

$$\frac{d\sigma}{d\Omega} = \frac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|} \times \int |\mathcal{M}|^2 \times \ \delta(E_1/c + m_2 c - z) \times |\mathbf{p}_3| \frac{dz}{z - |\mathbf{p}_1|\cos\theta}$$

Finish”

$$\frac{d\sigma}{d\Omega} = \frac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|} \times \int |\mathcal{M}|^2 \times \ \delta(E_1/c + m_2 c - z) \times |\mathbf{p}_3| \frac{dz}{z - |\mathbf{p}_1| \cos\theta}$$

- Final integral sets z in order to enforce energy conservation

$$z = E_1/c + m_2 c$$

$$\frac{d\sigma}{d\Omega} = \frac{S\hbar^2}{64\pi^2 m_2 c|\mathbf{p}_1|} |\mathcal{M}|^2 \frac{|\mathbf{p}_3|}{E_1/c + m_2 c - |\mathbf{p}_1| \cos\theta}$$