The Impulse Response of a Maxwell Earth

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An extended form of the correspondence principle is employed to determine directly the quasi-static deformation of viscoelastic earth models by mass loads applied to the surface. The stress-strain relation employed is that appropriate to a Maxwell medium. Most emphasis is placed on the discussion of spherically stratified self-gravitating earth models, although some consideration is given to the uniform elastic half space and to the uniform viscous sphere, since they determine certain limiting behaviors that are useful for interpretation and proper normalization of the general problem. Laplace transform domain solutions are obtained in the form of '*s* spectra' of a set of viscoelastic Love numbers. These Love numbers are defined in analogy with the equivalent elastic problem. An efficient technique is described for the inversion of these *s* spectra, and this technique is employed to produce sets of time dependent Love numbers for a series of illustrative earth models. These sets of time dependent Love numbers are combined to produce Green functions for the surface mass load boundary value problem. Through these impulse response functions, which are obtained for radial displacement, gravity anomaly, and tilt, a brief discussion is given of the approach to isostatic equilibrium. The response of the earth to an arbitrary quasi-static surface loading may be determined by evaluating a space-time convolution integral over the loaded region using these response functions.

CONTENTS

1. INTRODUCTION

Observations of slow changes in the earth's shape that occur in response to shifting surface loads provide important information concerning the rheological properties of the planetary interior. The inference of internal properties on the basis of such external observations is a good example of the kind of geophysical inverse problem that has been treated extensively by *Backus and Gilbert* [1967, 1968, 1970]. The solution of such inverse problems depends upon and is intrinsically limited by a physical model of the observed phenomenon. If there exist two different physical models both of which are capable of reproducing the existing observations (within experimental error), the inverse theory certainly cannot be expected to distinguish between them. If such a distinction is to be achieved, the data set must be extended, or the experimental error reduced, or both. This predicament, or at least a variant of it, may currently be the case with regard to the interpretation of the surface relaxation processes that are observed in the geological record to have accompanied Pleistocene deglaciation.

In the most widely accepted model for this postglacial uplift or rebound phenomenon the earth is taken to be a chemically homogeneous Newtonian viscous fluid or at most a linear viscoelastic solid. Such models have most recently been employed to interpret the observations by *Scheidegger* [1957, 1963] and *McConnell* [1963, 1965, 1968*a*, *b*]. These models are taken to have a single free parameter, the 'equivalent' Newtonian viscosity, and this parameter is assumed to be a function of radius only (in fact, virtually all such analyses have been concerned with plane earth models). Other parameters of the model (e.g., density, *P* and *S* wave velocities, etc.) are taken to be fixed by independent observations (e.g., free oscillation frequencies, body wave arrivals, etc.). Such subsurface models to explain the observed relaxation processes were first introduced by *Haskell* [1935, 1936, 1937] but had been studied earlier in a different context by *Darwin* [1879].

These fluid models of the observed rebound have recently been questioned by *Gjevik* [1972, 1973], who shows that if the subsurface thermal environment were favorable, then the surface could adjust to changes in the applied load by the radial migration of subsurface phase boundaries. Although 'favorable' conditions turn out to be a rather stringent restriction on this mechanism, it is also potentially capable of explaining the observed relaxation times. If the viscous fluid models were capable of adequately reproducing the observations, then the stringency of the required conditions for the phase transition process would be sufficient to eliminate the necessity of its consideration as an alternative, although it might nevertheless be desirable to incorporate it into a more general model. However, according to some authors [*Jeffreys*, 1940, 1970; *Magnitsky and Kalashnikova*, 1970; *Innes and Weston*, 1966] there is evidence of a distinct lack of correlation between the gravity anomaly and the magnitude of the vertical motion in regions such as Fennoscandia and Hudson Bay, which were centers for the largest Pleistocene ice sheets [*Paterson*, 1972]. More recent gravity data [*Gaposchkin and Lambeck*, 1971; *Kaula*, 1972] do not appear to support this lack of correlation. If gravity were the only driving force for

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the relaxation of the surface deformation, as it is assumed to be in the viscous fluid models, then it would be expected that such correlation would be close. The quantitative extent to which observed gravity anomalies differ from those that would be predicted by a fluid or linear viscoelastic model has unfortunately never properly been assessed. The purpose of this article is to explore a simple formalism within which it will subsequently be possible to provide the necessary assessment.

The basis for criticizing such estimates as have been produced of the difference between observed and predicted gravity anomalies is connected with the essentially global nature of the processes accompanying Pleistocene deglaciation. The main characteristic of the time dependent surface mass loads that result in readjustments of the earth's shape is that they correspond to load redistribution rather than simply to load removal. Not only did the glaciers melt, and the surface load thereby disappear from the region that it formerly occupied, but the ocean basins were also filled in the process. In the case of the Fennoscandian ice sheet, which had its energy (in the sense of a wave number spectrum of its thickness) concentrated near Legendre degree 15, one might expect that a plane earth approximation to the local uplift phenomenon (such as the approximations employed by *McConnell* [1965]) would be sufficient. However, simultaneously with the melting of the Fennoscandian glacier the so-called Laurentide ice sheet was also disappearing. This glacier, which covered all of Canada and parts of the United States, had a mean wavelength corresponding to a Legendre degree near 5. Consequently, its melting resulted in the application of a substantial load to the ocean basins. This load will clearly affect the relaxation process in Fennoscandia, but the magnitude of the effect has yet to be assessed. *Cathles*' [1971] analysis of the global problem has been the only attempt to provide a self-consistent treatment of the postglacial uplift problem in spherical geometry. If the fluid models of the uplift phenomenon are to be completely assessed, their theoretical framework must be cast into such a form that direct and unambiguous comparisons can be made with observation.

As was implied above, the impetus for extensive reanalysis of the relaxation process within the context of a fluid model derives primarily from the desire to identify the relaxation mechanism correctly. This can be done only by subjecting the currently accepted paradigm to stringent test. If the fluid model can be made to explain the data (within observational accuracy), then by implication the mantle should be taken to behave effectively as a Newtonian viscous fluid for deformations on time scales as short as 10,000 years. If it is correct, this implication has direct bearing upon the phenomena of continental drift and sea floor spreading, since the driving mechanism for these motions is currently believed to be associated with some form of thermal convection in the planetary mantle [*Tozer*, 1965; *Peltier*, 1972; *Richter*, 1973; *McKenzie et al.*, 1974] and since this phenomenon is strongly dependent upon fluid viscosity, a parameter determined by the analysis of uplift data. Since no attempt will be made in the present paper to infer a mantle viscosity profile and since this parameter is a fundamental ingredient in the viscoelastic models of the earth that are to be discussed, some rational basis is required for the utilization of the viscosity profiles selected for analysis in section 7.

Until the paper of *Goldreich and Toomre* [1969] it had commonly been believed that the viscosity of the lower mantle was considerably higher than the viscosity of the upper mantle. This belief was associated with the previously held belief [*Munk and MacDonald*, 1960; *MacDonald*, 1966] that there was 'something special' about the apparently excessive equatorial bulge of the earth and that this something was to be considered a fossil rotational bulge. Assuming an average value for the deceleration of the earth's rotation, Munk and MacDonald calculated that the equatorial bulge was equal in magnitude to the value that it should have had 9.5×10^6 years ago. *MacDonald* [1966] subsequently interpreted this value as implying a mantle viscosity of 7.9×10^{25} P. *McKenzie* [1966, 1967, 1968] came to the same conclusion by a different route and calculated a viscosity of 2.4×10^{27} P for the lower mantle.

All of these arguments were based upon the assumption that the gravitational energy in C_2^0 was larger than it was in the other spherical harmonic coefficients and that this was a significant characteristic of the earth. *Goldreich and Toomre* [1969] pointed out that the dominance of C_2^0 was due to a bias in the spherical harmonic description that overemphasized the energy in equatorial anomalies by a factor of 3. *O'Connell* [1971] gives sample computations that illustrate the effect on the spectrum of simple rotations of the coordinate axes. There is 'nothing special' about the equatorial potential bulge, and the earth is to be considered gravitationally triaxial. In conjunction with their polar-wandering model, *Goldreich and Toomre* [1969] then proceed to deduce a lower-mantle viscosity that is 'probably less than 6×10^{24} poise and certainly less than 6×10^{25} poise,' i.e., less by about 2 orders of magnitude than the value obtained by McKenzie.

An independent estimate of the viscosity of the lower mantle has been obtained by *Dicke* [1966, 1969] by making use of ancient eclipse records that allow extraction of the nontidal component of the acceleration of the earth's rotation. Combining these data with the assumption that the nontidal acceleration was produced by the filling of the oceans after deglaciation, assuming that the oceans are pure order 2 harmonic, and using an observed effective relaxation time for that harmonic (see original papers), Dicke calculated a value of about 10^{22} P for the viscosity of the lower mantle. After calculation on the full problem, *Cathles* [1971] suggested that this value should be adjusted to 3×10^{22} P. Following a similar line of attack, *O'Connell* [1971] obtains a further estimate of 6×10^{21} P, which is in accord with the last given value. His analysis is questionable, however, as will be discussed further in section 8.

From the above discussion it should be clear that there are several independent means by which the viscosity of the mantle may be inferred from surface observations. However, except for inferences based upon direct observations of the surface rebound after deglaciation, each of the methods discussed above carries with it the possibility of conceptual error as well as the usual numerical errors that are attendant upon any physical observation. A direct test of the fluid model should therefore involve a test of its detailed compatibility with the rebound observations. This paper is concerned with the first stage of such a test, namely, a description in theory of the way in which a general viscoelastic (Maxwell) model responds to a time variable surface mass load.

In order to make this discussion as generally applicable to the earth as possible it will be restricted to an analysis of the impulse response of the system. No attempt will be made to describe the response to a particular load. Given its impulse response, the response of a system to an arbitrary applied load can be obtained by evaluating a convolution integral over the surface of the sphere both in space and in time. This Green function approach has proved to be particularly successful in the analysis of the surface-loading problem for an elastic earth model [*Farrell*, 1972] and has been employed in a discussion of ocean tidal loading [*Farrell*, 1973]. The method was introduced originally by *Longman* [1962, 1963] in the same context as was employed by Farrell, but Longman was only able to compute the Green functions at distances sufficiently far from the source. The necessity for the introduction of this technique into the present context, where the problem is to describe the relaxation process associated with deglaciation, may not be immediately obvious. Normally, in seeking the response of a linear system to a forcing that has its energy concentrated in the low-degree spherical harmonics it is most efficient to evaluate the response in the wave number domain by multiplying the Love numbers (in analogy with the elastic problem) by the corresponding coefficients in the spherical harmonic expansion of the load. Examples of the application of this technique to the elastic problem are given by *Pertsev* [1966, 1970] and to a reduced viscoelastic problem by *Cathles* [1971]. The deglaciation problem is complicated, however, by several characteristics that suggest an alternative approach. First, the load has an extremely complicated temporal and spatial history; this fact is particularly true of the vast Laurentide ice sheet [*Paterson*, 1972]. Even more to the point in the case of the Laurentide region is a difficulty that would appear in attempting to compare observational data with predictions of the model. The data set is simply too sparse for a reliable spherical harmonic decomposition of it to be constructed [*Walcott*, 1972]. Both of these factors and others to be discussed in the concluding section suggest a Green function approach.

For the viscoelastic problem construction of Green functions would appear to be much more difficult than is found to be the case for the equivalent elastic system. This may be anticipated, if for no other reason, because the required Green function is time as well as space dependent. However, the anticipated increase in difficulty turns out to be spurious when the Maxwell constitutive relations are employed. For a Maxwell medium the correspondence principle is valid (true for any linear viscoelastic material). This principle says that the time dependent behavior of such a material may be obtained in the Laplace transform domain simply by solving an equivalent elastic problem for several values of the Laplace transform variables s. The time dependent behavior is then obtained by inverting the s spectrum so obtained. This is the point at which the analysis of such problems usually encounters difficulty. For the deglaciation Green functions, however, it turns out that numerical inversion of the transforms can be achieved in a relatively straightforward and elegant fashion. This approach to the construction of time histories and the formation of time dependent Green functions for vertical uplift, gravity anomaly, and tilt are the unique features of this review and extension of fluid theory describing the response of the earth to time dependent surface mass loads.

2. RESPONSE MECHANISM

The applicability of linear viscoelastic models to the description of relaxation processes in the mantle depends upon the physical details of the mechanism by which the system deforms under an applied stress. It is well known that the mode of material deformation depends not only upon the temperature and pressure conditions of the material but also upon the time scale over which the stress is applied. For example, although the terrestrial mantle must possess nonzero rigidity on short time scales ($\gtrsim$4 hours) on account of its seismically observed ability to transmit elastic shear waves, on time scales of the order of the earth's age (10^9 years) its behavior is well characterized by that of a viscous fluid (a material with zero rigidity). This follows from the fact that the theory of a rotating liquid spheroid [*Chandrasekhar*, 1970] is found to provide an accurate explanation of the oblate figure of the earth. In attempting to model the relaxation processes associated with deglaciation we are obliged to consider the response mechanism that is appropriate to deformation time scales of 'intermediate' length!

Materials that are at sufficiently high temperatures respond to suddenly applied stress with an immediate elastic deformation followed by a transient anelastic deformation called 'creep.' After a sufficiently long time the rate of deformation tends toward a constant value. This steady state creep controls the deformation in most high-temperature creep experiments on metals and ceramics. The pertinent experiments in this field and their geophysical implications have been reviewed by *McKenzie* [1968].

It is usually argued in analogy with these experiments that such steady state creep controls the phenomena of postglacial uplift of the crust and the associated motions in the mantle. The same argument is employed to justify the use of Newtonian fluid models of the thermal convection process supposed to be the driving mechanism for continental drift and sea floor spreading. Allowance is sometimes made for the possibility in the postglacial uplift problem that the response may have a significant elastic component. This has given rise to the introduction of linear viscoelastic models in this context, but such models have not been thoroughly discussed. The subject is a controversial one. *Weertman* [1970] has discussed the geophysical importance of experiment and theory for creep in pure materials and has shown that the Newtonian viscous fluid model serves to put a lower limit on the deformation rate under a given stress within the earth.

The equations of motion that are appropriate in describing the steady state deformation of a given medium clearly depend upon the creep mechanism. The two principal mechanisms found to be important in describing the experiments on ceramics and metals are diffusion creep [*Herring*, 1950] and dislocation climb [*Weertman*, 1955]. For diffusion creep there exists a linear relation between stress and strain, and so the deformation may be described through a viscosity ν, where

$$\nu = (kTa^2/\alpha D_0\Omega) \exp [(E + pV_a)/kT] \quad (1)$$

and where k is Boltzmann's constant, T the absolute temperature, V_a the activation volume, a the mean grain radius, E the activation energy for self-diffusion, p the pressure, Ω the atomic volume, and α a constant. In this case the equations of motion are easily derived and are the equations of viscous hydrodynamics with the temperature and pressure dependent viscosity (1).

It is usually assumed that diffusion is the creep-limiting process, for then it is appropriate to describe the relaxation of surface deformations following deglaciation in terms of a linear Stokesian viscous fluid model. This assumption is justifiable only if the internal stresses in the material are sufficiently small; this is a hypothesis that is not subject to direct refutation. It does not follow from the ability of a viscous fluid model to reproduce the observations that the creep-limiting process is diffusion. To verify this hypothesis, it would also have to be shown that nonlinear creep laws were incapable of producing similar agreement with the data.

In general, the contribution of the immediate elastic deformation to the total response must be included along with the viscous time dependent part. The simplest framework within which these two effects can be treated simultaneously is one in which the material is described by the constitutive relations appropriate to a Maxwell solid [*Eringen*, 1967]. For a Maxwell body with no initial stress the stress-strain relation is given by

$$\dot{\tau}_{kl} + (\mu/\nu)(\tau_{kl} - \tfrac{1}{3}\tau_{kk}\delta_{kl}) = 2\mu\dot{e}_{kl} + \lambda\dot{e}_{kk}\delta_{kl} \qquad (2)$$

where τ_{kl} and e_{kl} are the stress tensor and the strain tensor, respectively; the dot denotes time differentiation; δ_{kl} is the unit diagonal tensor; μ and λ are Lamé's constants; and ν is the viscosity. No previous study of the rebound problem has made full use of the Maxwell constitutive relation (2). The explicit dependence of ν upon the thermodynamic variables (p, T) has always been ignored; variations in ν are then incorporated as a pure space dependence by taking $\nu = \nu(r)$. The dependence of ν upon p and T in (1) is then invoked to imply that $\nu(r)$ must be a rapidly increasing function of depth and thereby to give credence to the old idea that the lower mantle must have a very high viscosity. This implication that diffusion creep would demand a large radial variation in viscosity is apparently erroneous (R. Smoluchowksi, personal communication, 1973) (see also the paper by *Weertman* [1970], whose analysis of the creep strength of the mantle from the point of view of solid state physics led to an estimated effective lower-mantle viscosity of 4×10^{22} P). The point of view adopted here is that the Lamé parameters in (2) are considered to be determined exactly by body wave and free oscillation data, so that the postglacial uplift observations refer directly to the $\nu(r)$ profile.

The assumption that ν varies only as a function of radius is clearly violated by the earth. The various scales of motion involved in the mantle general circulation, from the small-scale 'plumes' to the macroscale motions driving the lithospheric plates, are certainly characterized by lateral temperature variations. If the creep process is a thermally activated one, then these circulation systems must give rise to lateral variations of effective viscosity. The simplest example of a viscosity that is temperature dependent follows from the assumption of diffusion creep. In this case the dependence of ν on temperature is described through (1). If it were possible either by divine intuition or some other more prosaic means to determine precisely what are the lateral variations in planetary viscosity, then it would be possible to construct a spherical finite element model of the rebound into which this effect is incorporated. Numerical models of the mantle general circulation including the temperature dependence of viscosity will prove useful in providing initial estimates of the lateral variation of this quantity. For the present purposes we will ignore the possibility that such effects may be important, but in doing so we will point out that the two main centers of glaciation were located on stable platforms well removed from known centers of tectonic activity.

This model of the response mechanism, which consists of an immediate elastic plus a slow viscous reaction to load removal, ignores the possibility that effects that are entirely thermodynamic in origin might influence the relaxation process. For instance, the changing topography of the base of the lithosphere could interact with the mantle convection system. Such an interaction is unlikely to be significant, since the time scales of the two phenomena are so widely different. A more likely thermodynamic effect has been discussed by *Gjevik* [1972, 1973] and is dependent upon the fact [*Ringwood*, 1970, 1972; *Ringwood and Major*, 1970] that the mantle is chemically inhomogeneous.

Direct evidence for the existence of mantle phase transitions has in the past several years been provided by seismology. These phase transitions are marked by two 'strong' discontinuities in compression wave velocity at depths of 400 and 650 km and a possible third transition associated with a weak discontinuity near a depth of 1050 km. Recent advances in high-pressure technology [*Ringwood*, 1972] have enabled a correspondence to be drawn between the discontinuity at 400 km and the transformation of the mineral olivine to a more closely packed β spinel structure [*Ringwood*, 1970]. The approximately 230-kbar pressure at 650 km is currently inaccessible to direct experiment, but indirect methods (germinate analogs) have suggested that the β spinel phase should transform to a strontium-plumbate structure at pressures in this range.

The importance of these phase transitions in the mantle convection problem has been discussed by *Schubert and Turcotte* [1971] and *Peltier* [1971, 1972] within the framework of the linear stability theory by making use of a technique discussed by *Busse and Schubert* [1971]. The principal result of these calculations was the demonstration that the presence of the phase transitions actually serves to decrease the stability of the layer within the range of parameters that are likely to be typical of mantle conditions.

Gjevik [1972, 1973] discusses the possible importance of these phase transitions to the relaxation phenomenon that is observed to accompany deglaciation [*Walcott*, 1972]. The basic physical idea in this mechanism is that the phase transition must respond to a change in pressure at the earth's surface, since it takes place at fixed T, p conditions that are described by the Clapeyron curve. Owing to the density difference between the phases a motion must be induced in the surface as well as in the phase transition boundary. Unfortunately, the exact calculation of this effect is complicated, since the equations are nonlinear and such approximate solutions as have been obtained correspond to rather special cases and are of questionable applicability to the mantle. The most that can be said is that under the proper thermodynamic conditions the one-dimensional models that have been analyzed appear to be capable of giving relaxation times that are of the correct order to explain the glacial rebound data. Whether the mechanism is capable of explaining the observed geographical characteristics of the rebound, e.g., the peripheral bulge and the fact that some areas peripheral to the loaded region have undergone periods of emergence followed by periods of submergence, is, however, another question. These observations are consistent with the fluid model at least on the basis of the work reported here.

A more detailed analysis of Gjevik's mechanism would certainly be worthwhile. At the very least it must be said that the way in which phase transitions have been incorporated into previous rebound calculations as 'nonadiabatic density gradients' [*Cathles*, 1971] may be misleading at best.

However, before rejecting the simple Maxwell model on account of its linearity, its failure to incorporate phase transitions correctly, or any other reason it should first be determined quantitatively to what extent it is incapable of providing agreement with the observations.

3. CORRESPONDENCE PRINCIPLE

For a large class of problems in linear viscoelasticity the correspondence principle can be used to calculate a time dependent viscoelastic response from the solution to an 'associated' elastic problem. The basis for this rule is that with zero initial conditions the Laplace or Fourier time-transformed viscoelastic field equations and boundary conditions are formally identical with the equations for an elastic body of the same geometry. Thus transformed solutions can be calculated by standard elastic analysis and then inverted to obtain the time dependent response. This principle was deduced by *Lee* [1955] for isotropic media and later by *Biot* [1954, 1955*b*] for anisotropic materials. It will be extended here to include the case of nonzero initial stress. This is necessary in the context of the present problem, since previous to the onset of deglaciation the planet was of course prestressed hydrostatically. Biot has indicated that the principle is also applicable to variational methods of approximate elastic analysis. This fact has important implications for the mantle viscosity inverse problem. Furthermore, Lee observed that with 'proportional loading' (i.e., the space and time dependence of prescribed loads and displacements appear as separate factors with a common time factor) the spatial dependence of a transformed viscoelastic solution is the same as that which occurs in a geometrically similar elastic body if the spatial dependence of prescribed quantities is the same for both problems. A practical implication of this latter point is that with proportional loading, a transformed solution can be derived directly from an elastic solution by replacing elastic constants by operational moduli (compliances) and the time dependence of prescribed loading and displacements by transformed quantities.

For a Maxwell body the stress-strain relation (2) applies. To solve mechanical problems for which this relation is appropriate, the first step is to represent the tensors for stress and strain in terms of their Laplace transforms; thus (2) may be written as

$$\left(s + \frac{\mu}{\nu}\right)\tilde{\tau}_{kl} - \frac{1}{3}\frac{\mu}{\nu}\,\tilde{\tau}_{kk}\delta_{kl} = 2\mu s\tilde{e}_{kl} + \lambda s e_{kk}\delta_{kl} \tag{3}$$

where the tilde denotes the Laplace transform and s is the Laplace transform variable. Contracting the tensor relation (3) gives

$$\tilde{\tau}_{kk} = (3\lambda + 2\mu)\tilde{e}_{kk} \tag{4}$$

and substituting this result back into (3) then gives

$$\tilde{\tau}_{kl} = \left(\lambda + \frac{2}{3}\frac{\mu(\mu/\nu)}{(s + \mu/\nu)}\right)\tilde{e}_{kk}\delta_{kl} + \frac{2\mu s}{(s + \mu/\nu)}\,\tilde{e}_{kl} \tag{5}$$

Thus

$$\tilde{\tau}_{kl} = \lambda(s)\tilde{e}_{kk}\delta_{kl} + 2\mu(s)\tilde{e}_{kl} \tag{6}$$

where

$$\lambda(s) = \frac{\lambda s + \mu K/\nu}{s + \mu/\nu} \qquad K = \lambda + \tfrac{2}{3}\mu$$
$$\mu(s) = \mu s/(s + \mu/\nu) \tag{7}$$

Equation (6) has exactly the same form as the constitutive relation for a Hookean elastic solid, where the Lamé parameters $\lambda(s)$ and $\mu(s)$ are now functions of the Laplace transform variable s. The correspondence principle assures us that if we are willing to solve the equivalent elastic problem many times for different values of the Laplace transform variable s, then we will have constructed the Laplace transform of the time dependent viscoelastic solutions that we are seeking.

It is important to recognize that elastic analysis can only be used to calculate transformed solutions. The final step of inverting the transforms often proves to be extremely difficult if standard exact or asymptotic methods are used. This has been found to be particularly true (except for some elementary cases) when realistic material properties are employed. An example of the complexity that arises with the use of actual material properties can be seen in the paper by *Muki and Sternberg* [1961]. These difficulties have led several authors [e.g., *Lee and Rogers,* 1961] to reformulate the problems in terms of integral rather than differential equations, and in some cases this approach led to a relatively simple numerical scheme for calculating the time dependent solutions. However, since we are here interested in constructing solutions for full spherical self-gravitating earth models with inhomogeneous bulk properties, the integral equation approach appears to be hopeless at the outset. We are obliged to reconsider approximate methods for inverting the transform. Although many such methods have been proposed [*Bellman et al.*, 1966], I have found only one to be suitable to the particular problem of interest here. This method is due to *Schapery* [1962] and is discussed briefly in section 8.

4. BOUSSINESQ'S PROBLEM

Our intention is to treat the viscoelastic rebound problem by direct application of the correspondence principle. Since this treatment involves the construction of solutions to an equivalent set of elastic problems, the natural starting point for this discussion is the consideration of an appropriate set of such equivalent problems. The simplest such set is that considered by *Boussinesq* [1885] in his study of the response of a nongravitating elastic half space to an applied surface pressure. The main theoretical result was the determination of the Green function for the associated boundary value problem. This problem contains all of the essential ingredients of the more general spherical, self-gravitating, viscoelastic models without their associated numerical complexity. Furthermore, since the half-space response and that for the sphere converge at distances sufficiently near the point load (when it is assumed that the elastic forces dominate the gravitational forces in this range), the Boussinesq solution provides a convenient standard against which the spherical Green functions may be calibrated (see section 9).

a. Response of a homogeneous half space to a point force. When inertial forces are neglected, the Laplace-transformed displacement vector $\tilde{\mathbf{u}}$ must satisfy the equivalent elastic equilibrium equation throughout the medium. This may be written as

$$\sigma\nabla(\nabla\cdot\tilde{\mathbf{u}}) - \mu\nabla\times\nabla\times\tilde{\mathbf{u}} = 0 \tag{8}$$

where $\lambda(s)$ and $\mu(s)$ are the equivalent Lamé parameters defined in (7), $\sigma(s) = \lambda(s) + 2\mu(s)$, and $\eta(s) = \lambda(s) + \mu(s)$. In this section we follow the method of *Farrell* [1972], the only difference being in the interpretation of the final solution (section 4*b*). Suppose that the half space fills the region $z \le 0$. We seek a solution to (8) subject to the boundary condition that the surface $z \equiv 0$ is stress free everywhere except at the origin, where it is subject to a point force. We assume that this point force is applied to the surface only at time $t = 0$. Since the applied surface traction is a delta function in time, it must be equal to a constant in the Laplace transform domain wherein (8) is to be solved. By introducing a cylindrical coordinate system (z, r, θ) with basis vectors $\mathbf{e}_z$, $\mathbf{e}_r$, and $\mathbf{e}_\theta$ the solution to (8) may be represented as

$$\tilde{\mathbf{u}} = \tilde{u}(z, r)\mathbf{e}_z + \tilde{v}(z, r)\mathbf{e}_r \tag{9}$$

The θ dependence clearly vanishes by symmetry. We represent $\tilde{u}$ and $\tilde{v}$ in terms of Fourier-Bessel transforms of order 0 and 1, respectively, and denote by $\tilde{U}$ and $\tilde{V}$ the components of the transformed displacement. Thus

$$\tilde{u}(z, r) = \int_0^\infty \tilde{U}(z, \xi) J_0(\xi r)\xi \, d\xi$$
$$\tilde{v}(z, r) = \int_0^\infty \tilde{V}(z, \xi) J_1(\xi r)\xi \, d\xi \qquad (10)$$

where ξ is radial wave number. In (z, r, θ) coordinates the stress-strain relations are

$$\tilde{\tau}_{zz} = \sigma \, \partial\tilde{u}/\partial z + (\lambda/r) \, \partial/\partial r \, (r\tilde{v})$$
$$\tilde{\tau}_{rz} = \mu(\partial\tilde{v}/\partial z + \partial\tilde{u}/\partial r) \qquad (11)$$

The elements of the stress tensor $\tilde{\tau}_{ij}$ may also be expressed in terms of Fourier-Bessel transforms $(\tilde{T}_{zz}(x, \xi), \tilde{T}_{rz}(z, \xi))$ of order 0 and 1, respectively.

Transforming (8) and introducing the transforms of the stress-strain relation (11) give a coupled first-order system of equations that may be solved subject to appropriate boundary conditions on the surface $z = 0$. On this surface the tangential stress must vanish, so that

$$\tilde{\tau}_{rz}(0, r) = 0 \qquad (12)$$

For $\tilde{\tau}_{zz}$ we assume a stress arising from a unit force acting uniformly over a disk of radius α; then

$$\tilde{\tau}_{zz}(0, r) = -(1/\pi\alpha^2)H(s) \qquad r < \alpha$$
$$\tilde{\tau}_{zz}(0, r) = 0 \qquad r > \alpha \qquad (13)$$

where $H(s)$ is the Laplace transform domain representation of the time history of the applied load. Since we are interested in the impulse response of the system, we take

$$H(s) \equiv 1 \qquad (14)$$

This equation follows from the fact that the inverse Laplace transform of a constant is equal to a Dirac function that is the required time domain form of the applied load. In ξ space the stresses on $z = 0$ then have the representations

$$\tilde{\tau}_{zz}(0, \xi) = -(1/2\pi)\{2[J_1(\xi\alpha)/\xi\alpha]\}$$
$$\tilde{\tau}_{rz}(0, \xi) = 0 \qquad (15)$$

In the limit as $\alpha \to 0$, the disk load becomes a Dirac function in the space domain. This relationship can be seen from the fact that in this limit, $2J_1(\xi\alpha)/\xi\alpha \to 1$, and $\tilde{\tau}_{zz} \to -1/2\pi$, which is the transform of the Dirac function in cylindrical coordinates.

Boundary conditions (15) coupled with the requirement that both displacements and stresses must vanish in the limit $z \to -\infty$ enable solutions of the simultaneous set of equations to be constructed. Obtained in this way, the transformed displacements are

$$\begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} = -\frac{e^{\xi z}}{4\pi\mu\xi} \begin{bmatrix} (\sigma/\eta) - \xi z \\ (\mu/\eta) + \xi z \end{bmatrix} \qquad (16)$$

These displacements can be inverted by using (10) to give the fundamental solutions for the Laplace-transformed displacements caused by a unit normal point force on the surface of a homogeneous viscoelastic half space.

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{4\pi\mu R}\left(\dfrac{\sigma}{\eta} + \dfrac{z^2}{R^2}\right) \\ -\dfrac{1}{4\pi\eta R}\left(1 + \dfrac{z}{R} + \dfrac{\eta r^2 z}{\mu R^3}\right) \end{bmatrix} \qquad (17)$$

where $R^2 = r^2 + z^2$. If in (17) we take the limit $s \to \infty$, the parameters μ, σ, and η assume their elastic equivalents, as can be seen by inspection of (7). In this limit, (17) is just Boussinesq's solution (see the paper by *Farrell* [1972] for a more complete discussion of the elastic problem).

b. Laplace inverse of the elastic solution—Maxwell half space. Solutions (17) are the required s dependent solutions to the equivalent elastic problem. According to the correspondence principle these solutions are to be considered Laplace transform domain representations of the full time dependent solutions. Explicit use of (7) in (17) gives the following expression for $\tilde{u}(0, r)$:

$$\tilde{u}(0, r) = -\frac{1}{4\pi r}\cdot\frac{\sigma}{\mu\eta}$$
$$= \left[-\frac{1}{4\pi r}\right]\left[\frac{1}{\mu(s)} + [\lambda(s) + \mu(s)]\right]$$
$$= \left[-\frac{1}{4\pi r}\right]\left[\frac{s + \mu/\nu}{\mu s} + \frac{s + \mu/\nu}{(\lambda + \mu)s + \mu K/\nu}\right] \qquad (18)$$

This simple algebraic s spectrum can be inverted to give the exact time dependent solution for u (similarly, for v) as

$$u(0, r) = \int_L \tilde{u}(0, r)e^{st} \, ds \qquad (19)$$

where L is the Bromwich path. Thus

$$u(0, r) = \left[-\frac{1}{4\pi r}\right]\left[\frac{\sigma}{\mu\eta}\,\delta(t) + \frac{1}{\nu} + ce^{-t/\tau}\right] \qquad (20)$$

where

$$\tau = [\mu K/\nu\eta]^{-1}$$
$$c = (\mu/\eta\nu)[1 - (K/\eta)]$$

Equation (20) is the impulse response, or space-time Green function, for the vertical displacement of the surface of a homogeneous half space. The first term in (20) is the immediate elastic response of the system to the delta function load applied at $t = 0$. Its amplitude is just that obtained by Boussinesq. In the expression for the purely elastic solution ((17) in the limit $s \to \infty$) the $\delta(t)$ dependence is always implicit, since by definition a simple elastic medium adjusts instantaneously to an applied load if inertial forces are negligible. If inertial forces are not neglected, an elastic medium can of course support wave motions, in which case its 'adjustment' to the applied impulse takes an infinite length of time unless the system has finite Q (nonzero damping).

The second term in (20) is in a sense nonphysical; for suppose that we were to employ the Green function (20) to calculate the response of the half space to a point-step input (i.e., a constant point load applied at the origin at $t = 0$ and maintained). Convolution of the Green function with this load produces from the second term a term that grows linearly with t. Clearly, this is not an appropriate description of the response of the real earth, since the theory is correct only for infinitesimal strains. The source of this difficulty is to be found in the fact that we have failed to include gravitational restoring forces in the model. A full treatment of the physics including this effect will predict a steady (i.e., time independent) final state in which the gravitational and hydrodynamic forces are exactly in equilibrium. The mechanism by which this final state is attained is through gravitationally driven viscous flow in the half space—a transient convection produced by the loading.

This final state is the state of so-called isostatic equilibrium, to which we shall return for further discussion in section 10.

The last term in (20) describes the relaxation of the non-elastic part of the initial vertical surface deformation at a point caused by the applied impulse. For the homogeneous half space the decay constant τ is independent of the wavelength of the deformation. The magnitude of the amplitude parameter c is also independent of wavelength and measures the amount of vertical deflection that the surface will undergo (at position r) before equilibrium is attained. In the present example, in which gravitational forces have been neglected, there is of course no such equilibrium position possible.

For times that are sufficiently short after the initial application of a load the gravitational forces in the viscoelastic medium will still be negligible. For times sufficiently long after loading this is no longer a valid assumption. This can be seen by direct inspection of the operational compliances $\mu(s)$ and $\lambda(s)$ defined in (7). For large s, which corresponds to small t, $\mu(s) \to \mu$, $\lambda(s) \to \lambda$, and the Laplace transform domain solutions are exactly the elastic solutions discussed previously. The same limit is obtained by taking $\nu \to \infty$. In the limit $s \to 0$ ($t \to \infty$) the medium assumes the character of a Newtonian viscous fluid, and gravitational forces exert a dominant influence. This limiting behavior will be effectively obtained on a time scale that is proportional to ν. In the next section this regime is directly examined by neglecting elastic effects ab initio. In general, there will exist a 'transition time' for the response within which the elastic and viscous forces have comparable magnitudes (see section 8*b*).

5. Darwin's Problem

Boussinesq's problem was employed above in illustrating the analysis of a simple viscoelastic problem using the correspondence principle. In this section *Darwin*'s [1879] problem is similarly employed to introduce the concept of a 'decay spectrum' in describing the viscous relaxation of a small-amplitude deformation of the surface of a sphere. Explicit use is made of this concept in discussing the results of section 8.

Here we neglect the elastic forces entirely and make use of the stress-strain relation for a Newtonian viscous fluid to describe the deformation of a homogeneous spherical earth model. The equations of the model are the Navier-Stokes equations, the equation of continuity, and Poisson's equation. When it is assumed that the viscosity is uniform, the density is constant, and the medium is incompressible, then

$$\begin{aligned} \nu\nabla^2\mathbf{u} - \nabla_p + \rho\nabla\varphi &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \nabla^2\varphi &= -4\pi G\rho \end{aligned} \qquad (21)$$

where $\mathbf{u}$ is now the fluid velocity (not the parcel displacement, as is true in (8)), p is the mean normal stress (pressure), ρ is the density, φ is the gravitational potential, and G is the universal gravitational constant. Note that the coefficient ν in the hydrodynamic equations (21) represents the molecular viscosity. This is in accord with the usage of section 4 but is contrary to the custom in the hydrodynamics literature, where ν is reserved for the kinematic viscosity. If g is the surface gravitational acceleration, then $g = (4/3)\pi a\rho G$, and (21) may be nondimensionalized by the substitutions $\mathbf{u} = (ga)^{1/2}\mathbf{u}'$, $\varphi = ga\varphi'$, $\mathbf{r} = a\mathbf{r}'$, $p = ga\rho\pi$, and $t = (a/g)^{1/2}t'$. (This section is similar to *Parsons'* [1972] review of Darwin's work but has been modified to conform more closely to the general discussion in section 6.) When the primes are dropped, (21) becomes

$$\begin{aligned} \nabla^2\mathbf{u} + F\nabla(\varphi - \pi) &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned} \qquad (22a)$$

$$\begin{aligned} \nabla^2\varphi &= -3 \qquad r < 1 \\ \nabla^2\varphi &= 0 \qquad r > 1 \end{aligned} \qquad (22b)$$

where $F = \rho a(ga)^{1/2}/\nu$ is a nondimensional parameter. Consider that the fluid sphere has been distorted from perfect sphericity by gravitational interaction with a point mass load on its surface. By symmetry the distortion in shape will depend only upon the angular distance from the applied load. If R is the radius after such deformation, then

$$R = 1 + z' = 1 + \sum_{n=0}^{\infty} z_n P_n(\cos\theta) \qquad (23)$$

where $z' \ll 1$ is the deformation. The solution for the potential inside is

$$\varphi^i = \tfrac{1}{2}(3 - R^2) + \sum_{n=0}^{\infty} \varphi_n{}^i R^n P_n(\cos\theta) \qquad (24)$$

and the solution for φ outside is

$$\varphi^o = \frac{1}{R} + \sum_{n=0}^{\infty} \varphi_n{}^o R^{-n-1} P_n(\cos\theta) \qquad (25)$$

Now φ and $\nabla\varphi$ must be continuous across the deformed surface. When only first-order terms are kept in accord with the assumption $z' \ll 1$, there is the well-known result

$$\varphi_n{}^i = \varphi_n{}^o = [3/(2n + 1)]z_n \qquad (26)$$

The $\mathbf{u}$ is represented in terms of vector spherical harmonics as

$$\mathbf{u} = \sum_{n=0}^{\infty} \left(U_n(r)P_n(\cos\theta)\hat{e}_r + V_n(r)\frac{\partial P_n(\cos\theta)}{\partial\theta}\hat{e}_\theta \right) \qquad (27)$$

Substituting (27) into $\nabla\cdot\mathbf{u} = 0$ leads to

$$(1/R)\ \partial/\partial R\ (R^2U_n) = n(n + 1)V_n \qquad (28)$$

Taking the divergence of the momentum equation in (28) by using $\nabla\cdot\mathbf{u} = 0$ gives

$$\nabla^2(\varphi - \pi) = 0 \qquad (29)$$

Thus

$$\varphi - \pi = 1 + \sum_{n=0}^{\infty} c_n R^n P_n(\cos\theta) \qquad (30)$$

When (30) is substituted into (22), the radial part of the momentum equation is

$$\frac{\partial^2}{\partial R^2}(R^2U_n) - n(n + 1)U_n = -Fc_n nR^{n+1} \qquad (31)$$

A particular solution of the inhomogeneous equation (31) that is regular at the origin is

$$U_n = b_nR^{n-1} - [Fc_nnR^{n+1}/2(2n + 3)] \qquad (32)$$

The coefficients b_n and c_n are to be determined from the surface boundary conditions. The condition that the tangential stress must vanish on the deformed surface leads to

$$b_n = Fc_nn^2(n + 2)/2(2n + 3)(n + 1)(n - 1) \qquad (33)$$

The boundary condition on the normal component of the stress is that it should balance the surface load. In the present example the gravitational effect of the load is neglected. However, in the general viscoelastic problem discussed in the next section this effect will also be included. Since the potential on the deformed surface of the sphere is

$$\varphi(R = 1 + z') = 1 + \sum_{n=0}^{\infty} \varphi_n P_n(\cos\theta) \tag{34}$$

where

$$\varphi_n = [-2(n-1)/(2n+1)]z_n \tag{35}$$

Thus from (30)

$$\varphi_n - \pi_n = c_n \tag{36}$$

where the π_n are the coefficients in the spherical harmonic expansion of the pressure field π on the deformed surface. If the L_n are the coefficients in the spherical harmonic expansion of the load, then the boundary condition on normal stress gives

$$-L_n = -\pi_n + (2/F)(\partial U_n/\partial R) \qquad R = 1 \tag{37}$$

which when (33), (34), and (36) are used, gives

$$-L_n = \frac{2(n-1)}{(2n+1)} z_n + c_n \frac{(2n^2+4n+3)}{(2n+3)(n+1)} \tag{38}$$

Since the time derivative of the radial displacement must equal the radial component of velocity, then $dz_n/dt = (U_n)_{R=1}$, and from (38),

$$\frac{dz_n}{dt} = \frac{Fn(2n+1)}{2(n-1)(2n^2+4n+3)} \cdot \left[-L_n(t) - 2\frac{(n-1)}{(2n+1)} z_n(t)\right] \tag{39}$$

The case of free decay corresponds to $L_n = 0$ ($t > 0$ for delta function mass application), so that

$$dz_n/dt = -[Fn/(2n^2+4n+3)]z_n \tag{40}$$

or

$$z_n = z_n(0) \exp(-t/\tau_n) \tag{41}$$

where

$$\tau_n = (2n^2+4n+3)/Fn \tag{42}$$

Equation (41) is an important result that will help in understanding the solutions for the viscoelastic problems discussed in section 8. It states that each spherical harmonic component of the surface deformation of a sphere of uniform viscosity will decay exponentially with a decay constant that depends only upon the harmonic degree of the component (i.e., its wavelength).

In general, the solution of (39) may be written as

$$z_n(t) = z_n(0)\exp(-t/\tau_n) - \int_0^t \frac{F_n(2n+1)}{2(n-1)(2n^2+4n+3)} L_n(t') \cdot \exp[-(t-t')/\tau_n]\, dt' \tag{43}$$

(i.e., as a convolution of the load with the decay characteristics of the viscous sphere). Given the decay characteristics (decay spectrum) of the sphere, it is then possible to synthesize the response to an arbitrary time variable load. If the viscosity of the sphere and its density are not constant but are variable in the radial direction, there is no reason to expect that individual spherical harmonic components of the initial deformation will continue to relax in a characteristically exponential fashion. This assumption is often made nevertheless [*O'Connell*, 1971]. The extent to which this assumption is incorrect for realistic earth models is investigated in section 8, where in treating the viscoelastic problem the immediate elastic response may be isolated from the elasticoviscous transition plus pure viscous response to provide a means of comparison with the result (41). The result of this investigation is the direct refutation of the assumption of exponential relaxation. Depending upon the magnitude of the mantle viscosity the breakdown of this usual assumption may be traced either to the presence of the core or to the strength of the elasticoviscous transition. In either case the violation of the assumption is most striking for the low-degree harmonics.

6. Mass Loads on a Spherically Stratified Maxwell Earth

In section 4 the correspondence principle was employed to determine the response of a homogeneous viscoelastic half space to an applied point load. Here the same principle is used to calculate the response of general, spherically stratified earth models. In these calculations the gravitational effect of the load is included. The appropriate equivalent elastic problem is precisely that which must be solved in the calculation of elastic load tides. In this problem, solutions are sought to the same set of differential equations as are employed to describe body tides or free oscillations except for the fact that in the latter problem, inertial forces must be included. For these problems the set of differential equations are of course subject to different boundary conditions. A complete discussion of the spherical elastic earth problem (the equivalent problem for the viscoelastic analysis) is given by *Longman* [1962, 1963], *Takeuchi et al.* [1962], and *Kaula* [1963]. Only a brief sketch of the necessary analysis is given here, and this sketch follows *Farrell*'s [1972] extension and review of Longman's work.

a. Integration of the equations of motion. In the Laplace transform domain the equations of motion consist of the linearized equation of momentum conservation and Poisson's equation [*Backus*, 1967].

$$\begin{aligned} \nabla\cdot\tilde{\tau} - \nabla(\rho g\tilde{\mathbf{u}}\cdot\mathbf{e}_r) - \rho\nabla\tilde{\varphi} + g\nabla\cdot(\rho\tilde{\mathbf{u}})\mathbf{e}_r &= 0 \\ \nabla^2\tilde{\varphi} &= -4\pi G\nabla\cdot(\rho\tilde{\mathbf{u}}) \end{aligned} \tag{44}$$

Here ρ and g are the density and gravitational acceleration in the static equilibrium state, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\tau}$ is the stress tensor, and φ is the gravitational potential; φ is the sum of two parts φ_1 and φ_2, which are, respectively, the perturbation potential of the ambient gravitational field and the potential of the externally applied gravitational force field (the load). The tilde, which will be dropped hereafter, indicates that the quantity is a function of the Laplace transform variable s. The s dependence of (44) is contained in the form of $\tilde{\boldsymbol{\tau}}$ defined in (6). If we restrict attention to spherically symmetric earth models with a free outer surface and look for Laplace transform domain solutions for the deformation due to a point mass, then (44) reduces to the spheroidal system of equations in the three scalar variables u_r, u_θ, and φ. When $\mathbf{u}$ and φ are expanded in spherical vector harmonics as was done in section 5 and explicit use is made of the axial symmetry of the problem, then

$$\begin{aligned} \mathbf{u} &= \sum_{n=0}^{\infty}\left(U_n(r,s)P_n(\cos\theta)\mathbf{e}_r + V_n(r,s)\,\partial\frac{P_n(\cos\theta)}{\partial\theta}e_\theta\right) \\ \varphi &= \sum_{n=0}^{\infty}\varphi_n(r,s)P_n(\cos\theta) \end{aligned} \tag{45}$$

Three additional dependent variables are introduced: the radial and tangential components of the stress tensor, τ_{rr} and $\tau_{r\theta}$, and a variable q related to the radial gradient of potential and therefore to the ambient gravitational field as

$$q = \frac{\partial \varphi}{\partial r} + \frac{(n+1)}{r}\varphi + 4\pi G \rho u_r \qquad (46)$$

These additional variables are also written in terms of their Legendre transforms with coefficients $T_{r,n}$, $T_{\theta,n}$, and Q_n. Equations (44) then reduce to the matrix equation

$$d\mathbf{Y}/dr = \mathbf{AY} \qquad (47)$$

Where $\mathbf{Y} = (U_n, V_n, T_{rn}, T_{\theta n}, \Phi_n, Q_n)$ and $\mathbf{A}$ is the s dependent matrix given in the appendix; $\mathbf{A}$ reduces to the half-space limit when $\rho = 0$ and r and n are large.

Subject to appropriate boundary conditions, (47) may be solved by numerically integrating this set of six simultaneous ordinary differential equations over the range $r = 0$ to $r = a$ (where as was true before, a is the earth's radius). Two numerical schemes have been employed in constructing solutions to (47). The primary technique was the Runge-Kutta-Gill scheme [*Shanks,* 1966], with variable step size [*Backus and Gilbert,* 1968]. Selected solutions in critical regions were checked by using the 'predictor-corrector' method due to Hamming [*Ralston and Wilf,* 1960].

b. Boundary conditions. The numerical integration of (47) is begun at some starting radius r_0, below which it is assumed that the physical parameters λ, μ, ν, and ρ are independent of r. This depth, which in general is s dependent, is determined by trial and error; r_0 is chosen sufficiently small that further reduction leads to no change in the eigenfunction structure of the solution. For a uniform sphere there are analytic expressions for the linearly independent vectors $\mathbf{Y}$, which are finite at r_0. In a solid there are three such vectors, and in a fluid, two. These starting vectors, which are given by *Gilbert and Backus* [1968], may be propagated to the surface by either of the numerical integration schemes discussed above. The final solution is that mix of the three linearly independent solutions that satisfies the boundary conditions at the earth's surface.

When the Legendre degree was sufficiently low that the starting depth was in the core, then the core was assumed to have zero Brunt-Väisälä frequency (see section 6*c*). In fact, all solutions were found to be insensitive to the details of the core model used.

The boundary conditions at the surface have been discussed by *Longman* [1963] and more recently by *Farrell* [1972] for the quasi-static loading of an elastic sphere. In the present case great care must be taken with the surface boundary conditions, since these conditions are being applied in the Laplace transform domain. The time domain problem for which a solution is here being sought is for the deformation of a viscoelastic sphere when at $t = 0$ a point mass γ is brought up from infinity and instantaneously removed. We want to determine the time dependent shape and gravitational field of the sphere for $t \geq 0$. In order for the applied mass load to have the required delta function behavior in the time domain the boundary conditions in the Laplace transform domain must be s independent (see section 4*b*). The linearized boundary conditions are that $\tau_{rr}(a) = -g\gamma\delta(\theta)$ (the normal stress balances the applied load); $\tau_{r\theta}(a) = 0$ (the tangential stress is zero); and φ_1, φ_2, and $(\nabla\varphi_1 + 4\pi G\rho\mathbf{u}) \cdot \mathbf{e}_r$ are continuous, whereas $\mathbf{e}_r \cdot \nabla\varphi_2$ must change by $4\pi g\gamma$ across $r = a$. For a point mass load these conditions lead to

$$T_{r,n}(a) = -g(2n+1)/4\pi a^2$$
$$T_{\theta,n}(a) = 0 \qquad (48)$$
$$Q_n(a) = -4\pi G(2n+1)/4\pi a^2$$

As was stated above, these conditions are independent of s. They determine $U_n(r, s)$, $V_n(r, s)$, and $\Phi_n(r, s)$ for the point mass load in terms of which the Love numbers will be defined in section 7.

c. Treatment of the core. In the fluid core, which we assume to be inviscid, (44) reduces to a fourth-order system, since $\mu = 0$ and the tangential components of the displacement and stress can be eliminated. It was for a long time conventional in circumstances such as those considered here, where inertial forces are unimportant, to assume the so-called Adams-Williamson condition [*Longman,* 1963]. This condition asserts that the density stratification in the core must be adiabatic in order that a solution exist that satisfies the boundary conditions. *Smilie and Mansinha* [1971] rejected this postulate due to Longman by replacing the condition of neutral stratification of the core by one allowing for a discontinuous radial displacement at the core-mantle interface. *Dahlen* [1974] has recently given an extensive discussion of this point. *Pekeris and Accad* [1972] have also indicated the way in which the Adams-Williamson condition is to be circumvented. They determine solutions for the long-period bodily tides for core models having both stable and unstable as well as neutral density stratifications. With stable stratification an infinite number of core oscillations are found, as might have been expected from the outset, since under these conditions the medium will support Rossby gravity waves, in which the restoring force for parcel oscillations is a combination of Coriolis and buoyancy effects. Unfortunately, Pekeris and Accad did not include the Coriolis force in their work, so that their discussion of the low-frequency modes is incomplete. These waves are precisely the ones whose excitation gives rise to the thermal and gravitational tides in the earth's atmosphere. The Brunt-Väisälä frequency of the core is a subject of current contention [*Higgins and Kennedy,* 1971].

For the present purposes the question of the treatment of the core at zero frequency is avoided by assuming that the core is neutrally stratified.

d. Harmonics of degree 0 and 1. A load of the form P_0 is uniformly distributed over the entire earth's surface. Because of the finite compressibility of the medium this load causes a radial displacement that is accompanied neither by a tangential displacement nor by a perturbation in the gravitational potential. When $n = 0$, (47) reduces to a second-order system in U_0 and $T_{r,0}$ that has been given by *Longman* [1963]. Here of course the solutions are s dependent on account of (7). Since the glacial load (consisting of ice plus meltwater) conserves mass, the $n = 0$ coefficient of its Legendre decomposition must vanish identically. This does not mean that the U_0 term can be neglected in the Green function.

Displacements of degree 1 have normally been ignored, since they are accompanied by a shift in the center of mass of the earth. *Cathles* [1971] has observed that this is not correct for surface loading problems. Although it is true that the center of mass of the earth plus load is fixed in space, there is no constraint on the earth alone. *Farrell* [1972] has described a method of integrating the differential equations under this constraint when $n = 1$. It need not be described further here.

7. VISCOELASTIC EARTH MODELS

In the following sections several computations will be discussed in illustration of the effect of variations in viscoelastic structure on the response characteristics of the earth. Three such structures will be considered. These structures are all self-gravitating Maxwell models and so require the specification of ν as well as μ, λ, and ρ as functions of depth. In each model the Lamé constants μ and λ as well as the density ρ are taken to have the variations appropriate to the Gutenburg-Bullen model A earth, as described, for instance, by *Alterman et al.* [1961]. The three models analyzed then differ only in their dependence upon the effective Newtonian viscosity ν and are chosen to illustrate the extreme variations of this quantity, which have at times been suggested as being reasonable (see the remarks in section 1). These viscosity models are illustrated by Figure 1. Model 1 has a uniform viscosity of 10^{22} P throughout the mantle. Model 2 has a 300-km-thick surficial low-viscosity zone with a viscosity of 10^{21} P; otherwise, the viscosity is everywhere equal to that of model 1. Model 3 has a more viscous lower-mantle viscosity of 10^{24} P from a depth of 1000 km to the core-mantle boundary and an upper-mantle viscosity equal to that of model 1. In all models the core is taken to be essentially inviscid.

It should be noted that no attempt has been made to include the effect of the lithosphere in any of the models described above. This region can be simulated by a surface layer of high viscosity with a thickness according to current orthodoxy that is of the order of 100 km. Its presence will certainly alter the Green functions in the region close to the point source. The discussion of details like this will be left for future publications.

8. Time Dependent Love Numbers

In analogy with the problem of determining the static deformation of an elastic sphere, dimensionless Love numbers h_n, l_n, and k_n are introduced into the viscoelastic problem. These numbers are functions of three variables r, n, and s. If U_n, V_n, and $\Phi_{1,n}$ arise from a force field with potential φ_2 described through its coefficients $\varphi_{2,n}$, these Love numbers are defined by the following:

$$\begin{bmatrix} U_n(r, s) \\ V_n(r, s) \\ \Phi_{1,n}(r, s) \end{bmatrix} = \Phi_{2,n}(r) \begin{bmatrix} h_n(r, s)/g \\ l_n(r, s)/g \\ k_n(r, s) \end{bmatrix} \tag{49}$$

where $\Phi_{2,n}$ is independent of s, since the applied load is assumed to have a delta function dependence in the time domain. When attention is focused upon the solutions at $r = a$, then $\Phi_{2,n}(a) = ag/m_e$ [*Longman,* 1963], where g is the gravitational acceleration at the earth's surface and m_e is the earth's mass. The time dependence of the displacements **u** and the potential perturbation φ_1 is contained in the time dependence of the Love numbers. In the limit of large n and small θ the solutions of the spherical problem tend to the solutions for the half space (Boussinesq's problem, discussed in section 4). This equivalence leads to the asymptotic relations

$$\begin{bmatrix} h_n \\ nl_n \\ nk_n \end{bmatrix} = \frac{gm_e}{4\pi a^2 \eta} \begin{bmatrix} -\sigma/\mu \\ 1 \\ 3\rho\eta/2\langle\rho\rangle\mu \end{bmatrix} \tag{50}$$

which are correct to order $1/n$. In (50) the parameters η, σ, and μ have values equal to those for the top layer of the spherical earth models, and $\langle\rho\rangle$ is the earth's mean density. These asymptotic results were discussed in detail by *Farrell* [1972].

a. Love number s spectra and their inversion. Given the Love numbers defined in (49) and computed according to the methods discussed in section 6, the deformation of the sphere may be obtained in the space and time domains. The transformation from wave number (n) space to θ space is accomplished by summing the infinite series in (45). The transformation from the Laplace transform domain to the time domain is achieved by evaluating the inversion integral (19). In the present problem it matters very much which of the inversions is performed first. If the Legendre transform is evaluated before the Laplace transform, then the subsequent Laplace inversion is very difficult if not impossible to determine. Accordingly, we begin the discussion of numerical results and interpretation with the inversion of the Love number s spectra. The time dependent Love numbers are closely connected to the decay characteristics of the viscous sphere discussed in section 5 (Darwin's problem), as will be seen shortly. The discussion is restricted to the h_n and k_n Love numbers, since only these numbers are required in calculating the Green functions (see section 9) with which we are presently concerned.

The s spectra of Love numbers h_n and k_n are illustrated in Figures 2 and 3 for model 1 at selected values of n. The corresponding spectra for models 2 and 3 are included in Figure 11 (*a–d*). The shapes of these spectra are characteristic of those for a general relaxation process. The dynamic range of the spectrum for degree n is of the order of n, as could have been anticipated, and each is characterized by asymptotic regimes for sufficiently small and sufficiently large s.

These asymptotic regimes have direct physical interpretations. For large s, which corresponds to small t, the s dependent Love numbers approach their values for the purely elastic Gutenburg-Bullen A earth model. These large s Love numbers are precisely those computed earlier by *Kaula* [1963], *Kuo* [1969], and *Farrell* [1972]. The existence of the asymptotic regime for small s, which corresponds to large t, is connected with the existence of the state of isostatic equilibrium. Inspection of these spectra and application of the final value theorem for a Laplace transform, which states that

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s \cdot L[F(t)]$$

for arbitrary $F(t)$, indicate that as $t \to \infty$, both $h_n(t)$ and $k_n(t)$ tend to zero. This is simply mathematical confirmation of the anticipated result that for times sufficiently long after application of the point mass the earth returns to its initial ($t < 0$) state of zero deformation. For this problem the state of

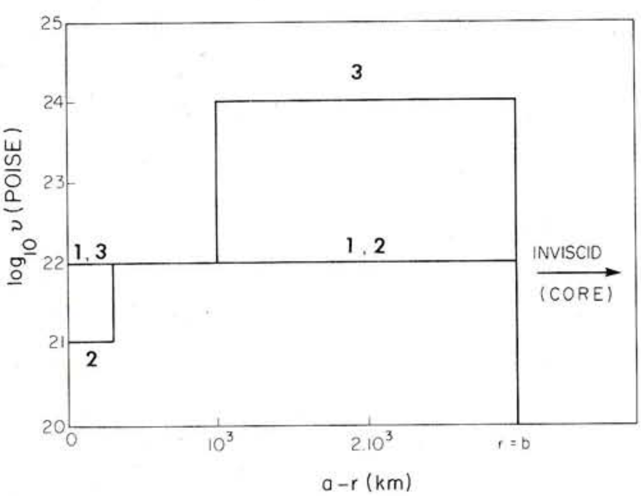

Fig. 1. Mantle viscosity models 1–3, described in text.

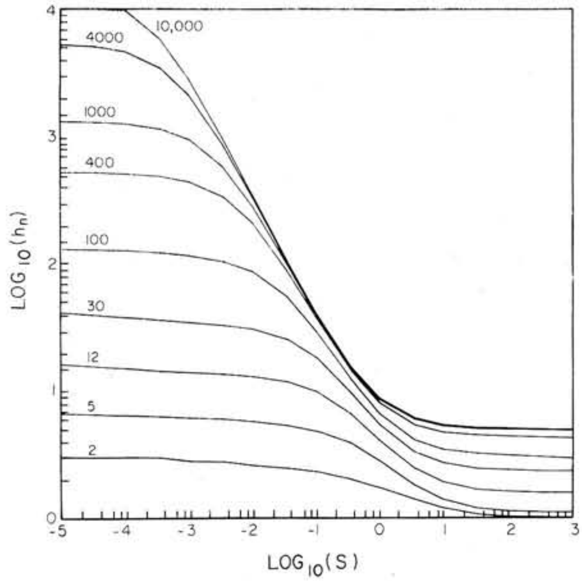

Fig. 2. Love number s spectra for viscosity model 1 (h_n). The corresponding value of n is marked beside each curve. Note the asymptotes for large and small s, which correspond to the elastic and isostatic limits, respectively.

isostatic equilibrium is the state of zero deformation. In general, this state is simply one in which a gravitational equilibrium exists between the deformed earth and a mass load on its surface (see section 10).

To facilitate the inversion of these s spectra into the time domain, it is convenient to write the h_n and k_n Love numbers in the form

$$h_n(s) = h_n^V(s) + h_n^E$$
$$k_n(s) = k_n^V(s) + k_n^E \tag{51}$$

where h_n^E and k_n^E are the constant large s asymptotes of each spectrum. When the Laplace inverse of (51) as specified by (19) is taken, then

$$h_n(t) = L^{-1}[h_n^V(s)] + h_n^E \delta(t)$$
$$k_n(t) = L^{-1}[k_n^V(s)] + k_n^E \delta(t) \tag{52}$$

Equations (52) emphasize that the large s asymptotes of the spectra in Figures 2 and 3 determine the immediate elastic response of the system. The weights h_n^E and k_n^E are precisely the surface mass load Love numbers that have been calculated for the elastic problem by previous authors. Equations (52) are analogs to (20) for the half-space problem. Both equations consist of an immediate elastic part plus a part containing the elasticoviscous transition and the long-term viscous behavior. In the spherical problem the latter part of the response is contained in the first term on the right-hand side of (52). It is to the inversion of this part of the s spectrum that we now turn.

The difficulty with these inversions arose from the fact that the $h_n^V(s)$ and $k_n^V(s)$ are known only numerically (i.e., with restricted accuracy) and for a limited number of real s values. The technique that we employ here to approximate the functions $h_n^V(t)$ and $k_n^V(t)$ is essentially the collocation technique described by *Schapery* [1962]. Under rather general assumptions *Schapery* [1962] has shown that the transient part of a viscoelastic response may be approximated by a Dirichlet series. If $\theta(s)$ is any one of the Love number s spectra, we assume that an exact inverse solution $\theta(t)$ exists in the form

$$\theta(t) = \sum_{i=1}^{m} a_i \exp(-t/\tau_i) \tag{53}$$

The motivation for choosing this form of an approximate solution is particularly apparent for the present problem, since we know that if the sphere had uniform density and viscosity, then the decay of each of the harmonic constituents of its deformation would decay as a pure exponential. Although we anticipate that the presence of the core and the intrinsically viscoelastic nature of the medium will result in deviations from pure exponential relaxation, we nevertheless expect that it will be possible to approximate the time history of each harmonic by a characteristic distribution function of relaxation times. In (53) the coefficients a_i, $i = 1, m$, are in fact a discrete approximation to such a distribution function.

To determine the a_i and τ_i in (53), we minimize the mean square error between $\theta(t)$ and an approximation to $\theta(t)$ that we call $\theta*(t)$, where

$$\theta*(t) = \sum_{i=1}^{m} b_i \exp(-t/\alpha_i) \tag{54}$$

The mean square error between $\theta(t)$ and $\theta*(t)$ is

$$E^2 = \int_0^\infty [\theta(t) - \theta*(t)]^2 \, dt \tag{55}$$

We assume that the α_i are fixed such that $1/\alpha_i = s_i$, where the s_i are the values of s at which we have computed the spectral amplitude. The b_i of the approximate solution (54) may then be determined by minimizing E^2 in (55) with respect to the b_i as

$$\frac{\partial E^2}{\partial b_i} = \int_0^\infty 2[\theta(t) - \theta*(t)] \exp\left(\frac{-t}{\alpha_i}\right) dt = 0 \tag{56}$$

Thus

$$\int_0^\infty \theta(t) \exp\left(\frac{-t}{\alpha_i}\right) dt = \int_0^\infty \theta*(t) \exp\left(\frac{-t}{\alpha_i}\right) dt \tag{57}$$

in order for the mean square error of the approximation to be a minimum, and thus the Laplace transform of the approximation must equal the Laplace transform of the exact solution at

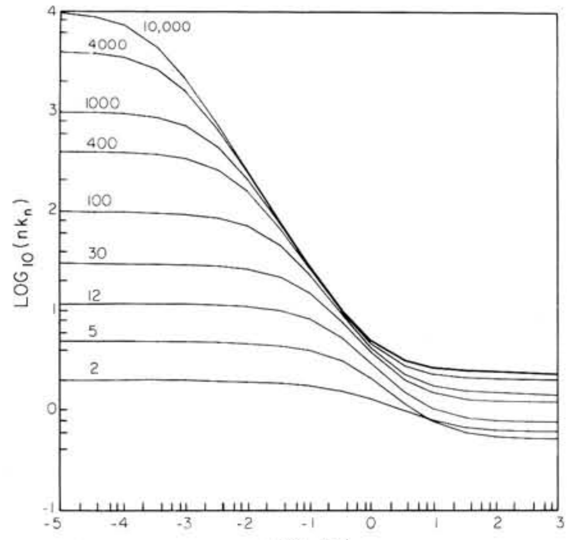

Fig. 3. Love number s spectra for viscosity model 1 (nk_n). Note the intersection of the small n spectra at intermediate s values.

least at the m points $s = 1/\alpha_i$, $i = 1, m$, or

$$\theta(s)|_{s=1/\alpha_i} = \theta_*\ (s)|_{s=1/\alpha_i} \quad (58)$$

implying that

$$\theta(s)|_{s=1/\alpha_i} = \sum_{j=1}^{m} \frac{b_j}{[s + 1/\alpha_j]}\bigg|_{s=1/\alpha_i} = \sum_{j=1}^{m} \frac{b_j}{[1/\alpha_i + 1/\alpha_j]} \quad (59)$$

a result that follows from the fact that $L(e^{-at}) = 1/(s + a)$. Equation (59) may be written in matrix form as

$$\theta_i = m_{ij} b_j \quad (60)$$

where $m_{ij} = \alpha_i\alpha_j/(\alpha_i + \alpha_j)$. Given the b_j obtained by solving the set of simultaneous equations (60), these may be employed in the Dirichlet series (54) to approximate the time history $\theta(t)$. The sequence b_j may be thought of as a discrete approximation to the distribution function of relaxation times as mentioned above. In practice, stable time histories may be obtained from samples of the s spectra taken twice per decade in s and confined to the portions of the curves in Figures 2 and 3 wherein the greatest rates of change of spectral amplitude are located. The linear algebraic system (60) is solved for the b_j by using standard techniques.

b. Love number time histories. In Figures 4, 5, and 6, examples of time dependent Love numbers h_n are shown for models 1, 2, and 3, respectively. The corresponding curves for the k_n are included in Figure 12(a–c). These curves were obtained from their corresponding s spectra by using the collocation technique discussed in section 8a above. All plots are log linear to emphasize that most of the large n Love numbers do decay exponentially (at least for sufficiently large t) for all three models. There are several important characteristics of these decay spectra to which attention should be drawn.

The first fact that should be recognized is that these time histories are for free decay. There is no load on the surface for $t > 0$. The intercepts of the curves for $t = 0$ correspond to the coefficients of the Legendre expansion of the anelastic part of the deformation after the immediate elastic response and recovery have taken place. This anelastic deformation proceeds to decay, partly by a mixture of elastic and viscous effects (elasticoviscous transition) and partly through the establishment of a transient viscous convection in the planetary interior. The viscous flow mechanism increasingly dominates the response at greater times after load removal. Only the decay curves for model 3 clearly exhibit the first stage of the decay process, in which the transition from elastic to viscous response occurs. Here there is an initial time interval with a duration much less than 1000 years during which the response is anomalously fast in comparison with the rate of decay in the later stages of adjustment. This is the elasticoviscous transition. The duration of this stage depends upon the relative magnitudes of the elastic and viscous parameters. When the elastic parameters are kept fixed, increases of viscosity result in increased duration of the transition region. This explains why the transition is more obvious in the response of model 3, since for this model the viscosity of the lower two thirds of the mantle is increased by a factor of 10^2 over its value in models 1 and 2. The separation (51) of Love numbers into viscous and elastic parts is therefore not a complete separation. The viscous part still contains a remnant elastic influence.

A second fact that should be recognized is clear on inspection of the decay curves for small degree in models 1 and 2. These small-degree deformation coefficients decay in a strikingly nonexponential manner. This is an important point, since it has often been assumed in analogy with Darwin's problem (section 5) that all harmonics will decay in a characteristically exponential fashion. For instance, *O'Connell* [1971] makes direct use of this assumption for the decay of degree 2 to construct an argument using data on the nontidal acceleration of the earth's rotation that allows him to estimate the viscosity of the lower mantle. Since his assumption was incorrect, his derived viscosity is certainly incorrect also. Determination of the size of the error will involve use of the accurate calculations given herein. The reason for the strongly nonexponential style of decay of the low-degree harmonics is of course the presence of the core. Successively higher degree harmonics are less influenced by its presence and so tend to behave analogously to the prediction of Darwin's problem. For model 3 the situation is somewhat different. Because of the

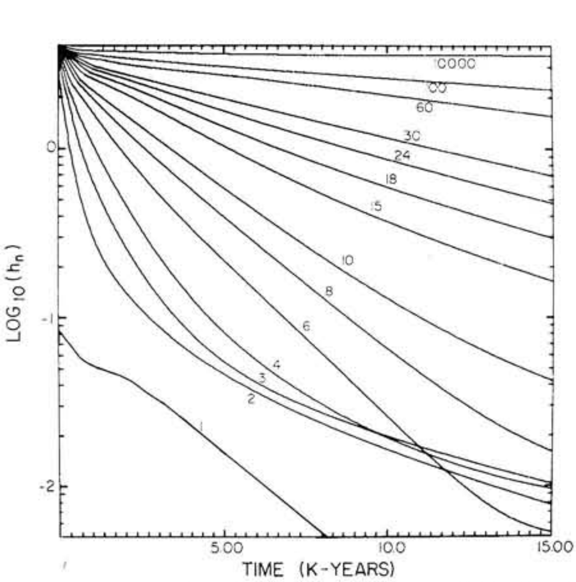

Fig. 4. Time dependent Love numbers h_n for model 1. The plot is log linear. Note the strongly nonexponential style of relaxation of the low-order Love numbers and the extremely slow exponential style of decay of the n = 10,000 Love number.

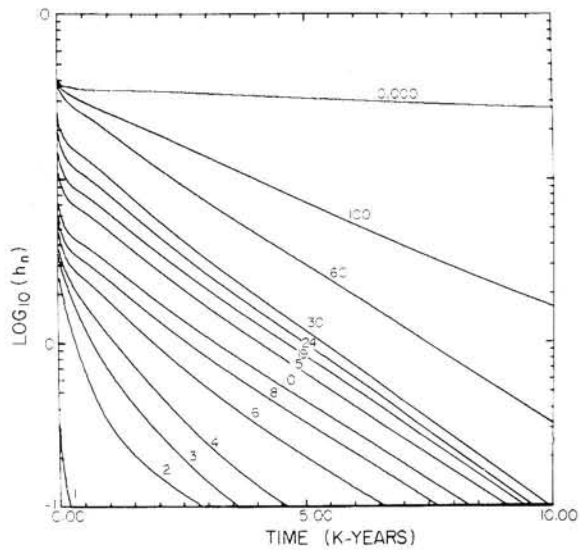

Fig. 5. Time dependent Love numbers h_n for model 2. Note the much faster rate of decay of the intermediate-order Love numbers (e.g., n = 100) than for model 1 (Figure 4). This is the effect of the low-viscosity zone.

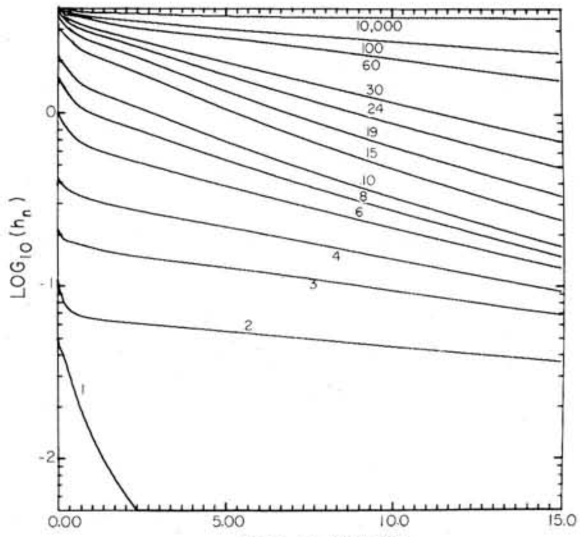

Fig. 6. Time dependent Love numbers h_n for model 3. Note the much slower rate of decay of the low-order Love numbers than for models 1 and 2. This is the effect of the hard lower mantle.

viscous lower mantle in this model, even the lowest-degree spherical harmonics are not noticeably affected by the presence of the core. The transient convection is confined to the upper-mantle region. Nevertheless, the low-degree harmonics in the decay spectrum for this model have a strongly nonexponential decay history for the first few hundred years after removal of the load. This is just the time interval over which the transition from elastic to viscous behavior is taking place (as was discussed above). After the transition has been completed, the decay of all harmonics is essentially exponential for this model, and the analogy with Darwin's problem is correct.

The discussion of the last paragraphs can be summarized as follows: in real earth models, individual spherical harmonic components of a surface deformation may exhibit a nonexponential free decay time history for two main reasons. If the viscosity of the mantle is sufficiently low, then all low-degree harmonics will 'see' the core and will decay nonexponentially. On the other hand, if the viscosity of the mantle is sufficiently high, then the elasticoviscous transition will be strong and will lead again to a nonexponential style of decay immediately following removal of the load.

A third point that should be made here concerns the information distribution as a function of wave number, which is contained in the Love number time histories. The eigenfunctions associated with individual Love numbers have their energies concentrated in progressively shallower depths as the degree increases. The fact that only the lowest-degree Love numbers are at all affected by the lower mantle and core is revealed in the way that the starting depth varies as a function of n for the numerical integration of (47). Information on the shallow structure of the model is contained in the decay curves of high degree, and information on the deep structure of the model, in the decay curves of low degree. For example, the low-viscosity zone in model 2 results in an increased decay rate for the high degrees but in only a weak effect for the low degrees. The viscous lower mantle in model 3 leads to a striking reduction in the rates of decay of the low degrees but has only a slight effect on the high degrees.

Degrees 0 and 1, which are not shown clearly on these time histories (degree 1 is just visible), both show time domain behavior that is delta function–like. Their amplitudes decrease very quickly after removal of the load.

9. SPACE-TIME GREEN FUNCTIONS

a. General discussion. Impulse response, or Green, functions are obtained for the surface mass load boundary value problem by summing infinite series of the form of (45). By means of (45), (49), and (52) the radial displacement can be written as

$$u_r(\theta, t) = \frac{a}{m_e} \sum_{n=0}^{\infty} [h_n{}^V(t) + h_n{}^E \delta(t)] P_n(\cos \theta) \qquad (61)$$

Similar expressions can be obtained for the perturbation in the gravitational field and for the deflection of the vertical. In both of these effects, which are referred to as the gravity anomaly and the tilt, the direct attraction of the mass load is important. There are also horizontal and vertical accelerations from the perturbed density field proportional to the Love number k_n. A third contribution to the gravitational perturbation is the change in acceleration produced by moving through the gradient in the unperturbed gravity field. The tilt is also influenced by a third effect due to the tilt of the deformed boundary. Combining all of these effects, *Longman* [1963] obtains expressions that have the following viscoelastic counterparts:

$$g(\theta, t) = \frac{g}{m_e} \sum_{n=0}^{\infty} \{ n\delta(t) + 2[h_n{}^V(t) + h_n{}^E \delta(t)] - (n + 1)[k_n{}^V(t) + k_n{}^E \delta(t)] \} P_n(\cos \theta) \qquad (62a)$$

$$t'(\theta, t) = \frac{-1}{m_e} \sum_{n=0}^{\infty} \{ \delta(t) + [k_n{}^V(t) + k_n{}^E \delta(t)] - [h_n{}^V(t) + h_n{}^E \delta(t)] \} \frac{\partial P_n(\cos \theta)}{\partial \theta} \qquad (62b)$$

which depend only upon h_n and k_n. The physical fields u_r, g, and t' defined above are the parameters that are capable of measurement geologically, and they are therefore the only parameters necessary for comparison of the theory with observation. (A further useful field is the potential perturbation ϕ, defined as $\phi(\theta, t) = (ag/m_e)\sum_{n=0}(1 + k_n - h_n)P_n(\cos \theta)$. It will be discussed elsewhere.)

Each of these expressions may be split into an elastic and a viscous part. For instance, (60) becomes

$$u_r(\theta, t) = \left[\frac{a}{m_e} \sum_{n=0}^{\infty} h_n{}^E P_n(\cos \theta) \right] \delta(t) + \frac{a}{m_e} \sum_{n=0}^{\infty} h_n{}^V(t) P_n(\cos \theta) = u_r{}^E(\theta)\delta(t) + u_r{}^V(\theta, t) \qquad (63)$$

The part with time dependence $\delta(t)$ is just the elastic Green function calculated by *Farrell* [1972]. It will not be discussed further here. The viscous parts of the impulse response functions for radial displacement, gravity anomaly, and tilt are then

$$u_r{}^V(\theta, t) = \frac{a}{m_e} \sum_{n=0}^{\infty} h_n{}^V(t) P_n(\cos \theta) \qquad (64)$$

$$g^V(\theta, t) = \frac{g}{m_e} \sum_{n=0}^{\infty} [2h_n{}^V(t) - (n + 1)k_n{}^V(t) P_n(\cos \theta)] \qquad (65)$$

$$t^V(\theta, t) = \frac{-1}{m_e} \sum_{n=0}^{\infty} \left(k_n{}^V(t) - h_n{}^V(t) \frac{\partial P_n(\cos \theta)}{\partial \theta} \right) \qquad (66)$$

Several tricks can be employed to sum these series, which as they stand, are very slowly converging. Inspection of any of the Love number time histories discussed in section 8 (Figures 4–6) indicates that for fixed t (time) there exists a collection point toward which as n increases, each of the Love numbers tends. This is to be expected physically, since the decay rate becomes infinitely slow as $n \to \infty$. When these limiting values of the Love numbers are denoted by $h_\infty{}^V(t)$ and $k_\infty{}^V(t)$, the sums (65)–(66) are further separated with the substitutions

$$\begin{aligned} h_n{}^V(t) &= [h_n{}^V(t) - h_\infty{}^V(t)] + h_\infty{}^V(t) \\ k_n{}^V(t) &= [k_n{}^V(t) - k_\infty{}^V(t)] + k_\infty{}^V(t) \end{aligned} \qquad (67)$$

When (67) is substituted into (64), then

$$u_r{}^V(\theta, t) = \frac{a h_\infty{}^V(t)}{2m_e \sin(\theta/2)} + \frac{a}{m_e} \sum_{n=0}^{\infty} [h_n{}^V(t) - h_\infty{}^V(t)] P_n(\cos\theta) \qquad (68)$$

since $\sum_{n=0}{}^\infty P_n(\cos\theta) = \frac{1}{2}\sin(\theta/2)$. The remaining infinite sum in (68) will terminate after some $n = N$, since as was mentioned above, $h_n{}^V(t) \to h_\infty{}^V(t)$ as $n \to \infty$. This is called Kummers' transformation. Two further alterations are necessary in order to put the infinite series into a readily summable form. The first is to include a disk factor in the kernel of the sums like (68), so that the point mass is approximated by a mass that is spread over a small but finite area. This effectively clips the amplitude of the large n terms. The second further alteration is to transform the infinite sums by means of the Euler transformation [*Hildebrand,* 1956] into alternating series. The numerical application of this transformation to series like (68) is facilitated by the van Wijngaarden algorithm. These problems have been discussed by *Farrell* [1972] within the context of the elastic problem. Solution of the viscoelastic problem through the correspondence principle enables most of this formalism to be transplanted intact. The only essentially new numerical ingredient is a method of doing the inverse Laplace transformation to get the $h_n{}^V(t)$ and the $k_n{}^V(t)$ from the $h_n{}^V(s)$ and the $k_n{}^V(s)$. Such a method was discussed in section 8.

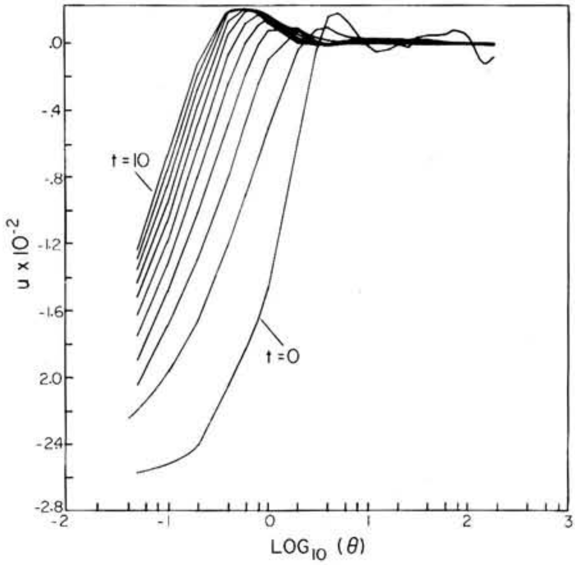

Fig. 8. Viscous part of the Green function for radial displacement in model 2. Time slices are at 1000-year intervals. Note the complex deformation of the peripheral bulge as it migrates inward.

b. Radial displacement, gravity anomaly, and tilt. The 'viscous' parts of the Green functions for radial displacement in models 1–3 are shown in Figures 7–9. The corresponding response functions for gravity anomaly are included in Figure 13(*a–c*). At near and intermediate distances from the point of application of the load ($\theta = 0$) the Love number $h_n{}^V$ dominates the anelastic part of the gravity anomaly, so that except near the antipode ($\theta = 180°$) this response function is similar to radial displacement.

The displacement Green functions have been normalized to the response

$$u_{\text{norm}}(\theta) = (-g\sigma/4\pi\mu\eta)(a\theta) \qquad (69)$$

which from (17) is seen to be just that of a homogeneous elastic earth model with $R = (a\theta)$. The parameters σ, η, and μ are taken as those appropriate to the uppermost layer in the stratification. On the other hand, the Green function for

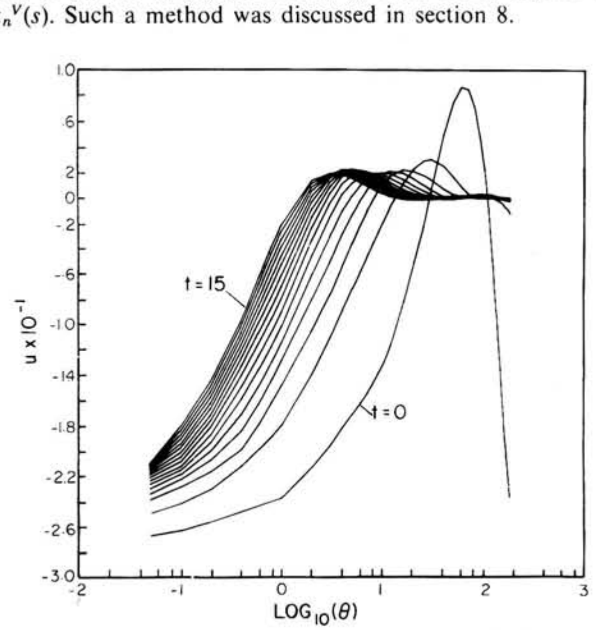

Fig. 7. Viscous part of the Green function for radial displacement in model 1. Time slices are at 1000-year intervals as shown. Note the inward migration of the peripheral bulge and the fast relaxation of the antipodal ($\theta = 180°$) depression.

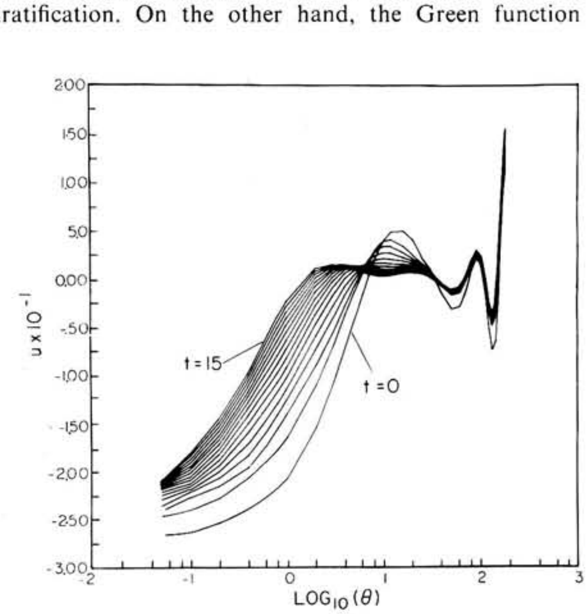

Fig. 9. Viscous part of the Green function for radial displacement in model 3. Time slices are at 1000-year intervals. Note the stationary nature of the collapsing peripheral bulge and the absence of relaxation at the antipode.

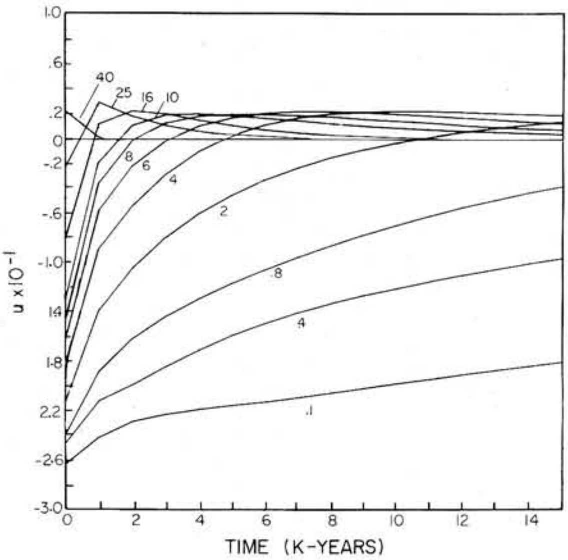

Fig. 10. Fixed θ slices through the Green function for radial displacement of model 1 (Figure 7). Note the existence of a range of θ, wherein the vertical motion is not monotonic owing to the migration of the peripheral bulge.

gravity anomaly is normalized with respect to the direct attraction of the mass load. This is just the part

$$g_{\text{norm}}(\theta) = \frac{g}{m_e} \sum_{n=0}^{\infty} nP_n(\cos\theta)$$

$$= \frac{-g}{4m_e \sin(\theta/2)} \quad (70)$$

of the infinite series (61). The normalizations are the same as those employed by *Farrell* [1972] and are convenient for the viscous as well as the elastic part of the Green functions. The normalized Green functions for the elastic and anelastic parts of the response are thus amenable to direct intracomparison.

Figures 7, 8, and 9 illustrate the $u_r^V(\theta, t)$ functions for a succession of times spaced at 1000-year intervals in the decay histories of models 1, 2, and 3, respectively. Foremost among the several noteworthy features of these response functions is the clear presence in all cases of a peripheral bulge at intermediate distances from the point of application of the load. Near the load the surface of the earth is inevitably depressed below its equilibrium level. Further away from the load is a region that has been uplifted above this equilibrium level. At still greater distances the deformation is strongly model dependent and may have either an antipodal depression as there is in model 1 or an antipodal elevation as there is in model 3.

This determination of the structure of the region of uplift surrounding the central depression and of its time history for an impulsively applied point mass load is among the most interesting results to emerge from the present initial study. The concept of this forebulge region was introduced by *Jamieson* [1882] and has been discussed by *Daly* [1920], *Walcott* [1970], and other authors. In the viscoelastic problems under analysis here the peripheral bulge is shown to be almost entirely due to anelastic effects, it having only an extremely modest expression in the elastic part of the Green function [*Farrell,* 1972].

Although the existence of the bulge is independent of the viscosity model, its position in space and its evolution in time certainly are not. The Green functions indicate precisely how strongly dependent upon the viscosity structure are the styles of relaxation at intermediate and long range. The styles of decay of the peripheral bulge for models 1–3 are strikingly dissimilar. The most prominent features of the bulge are its width in degrees, its nondimensional amplitude, and the location of its amplitude maximum. Figure 7, which illustrates the decay of the surface deformation for model 1, in which the mantle viscosity is uniform, has a bulge that migrates rapidly inward as a function of time. As the bulge moves inward, its amplitude at first decreases rapidly but after 2000 years remains nearly constant with a nondimensional amplitude of about +2 units. This inward migration is accompanied by a modest increase in width.

The behavior of the forebulge in model 2 (Figure 8), which has a 300-km-thick low-viscosity zone, is distinctly different from that in model 1. In this case the $t = 0$ location of the bulge is much closer to the source than was the case for model 1. Although the inward migration of the bulge is still a dominant characteristic of the response, the rate of inward migration is reduced from that for model 1. In addition, the amplitude of the bulge first decreases, as was true previously, but later increases again rather rapidly. This whole process is accompanied by a marked increase in the width of the forebulge as the decay proceeds. Physically, these processes may be partly understood on the basis of the tendency of the low-viscosity zone to force adjustment to proceed by flow essentially confined to a channel. Of course, such confinement is easier to achieve for the short-wavelength than for the long-wavelength components of the original deformation. Since the long wavelengths decay rather quickly for this model (see section 8), the later times in the decay history are times when the approximation to a channel flow is most closely achieved.

For model 3 (Figure 9) the characteristics of the relaxation process again differ markedly from those of either of the first two models. For this model the high viscosity of the lower 2000 km of the mantle severely reduces the decay rates of the lower-order harmonic components of the initial deformation (see section 8). This reduction results in an antipodal deformation that is very nearly time independent, since for large θ the dominant terms in the infinite series (64) are those with small n. This behavior of model 3 should be compared with the opposite behavior of models 1 and 2, for which the antipodal deformation has essentially been erased within the first 1000 years after removal of the load. This immediately suggests that if it were possible to obtain land emergence measurements in the antipodal region of a Pleistocene deglaciation, then one could interpret this information directly in terms of the viscosity of the lower mantle. The implementation of this idea will of course require careful attention to the ocean-filling part of the global rebound convolutions. Aside from the near time independence of the antipodal deformation the relaxation of model 3 is characterized by the fact that its peripheral bulge does not migrate. Although the initial forebulge is more than twice as large as that for model 1, as time proceeds, this deformation of the surface simply collapses in situ with only moderate change of width.

On the basis of the above discussion of the relaxation characteristics of models 1–3 it should be clear that relaxation data are eminently suited to the task of discriminating between different mantle viscosity structures. This of course assumes (as was made clear in the introduction) that the response mechanism is essentially a viscous hydrodynamic one. The main data set for the rebound problem (e.g., that of *Walcott* [1972] for the Laurentide region) consists of time dependent elevations as a function of position (referred of course to a particular and assumed known eustatic sea level curve). Near the

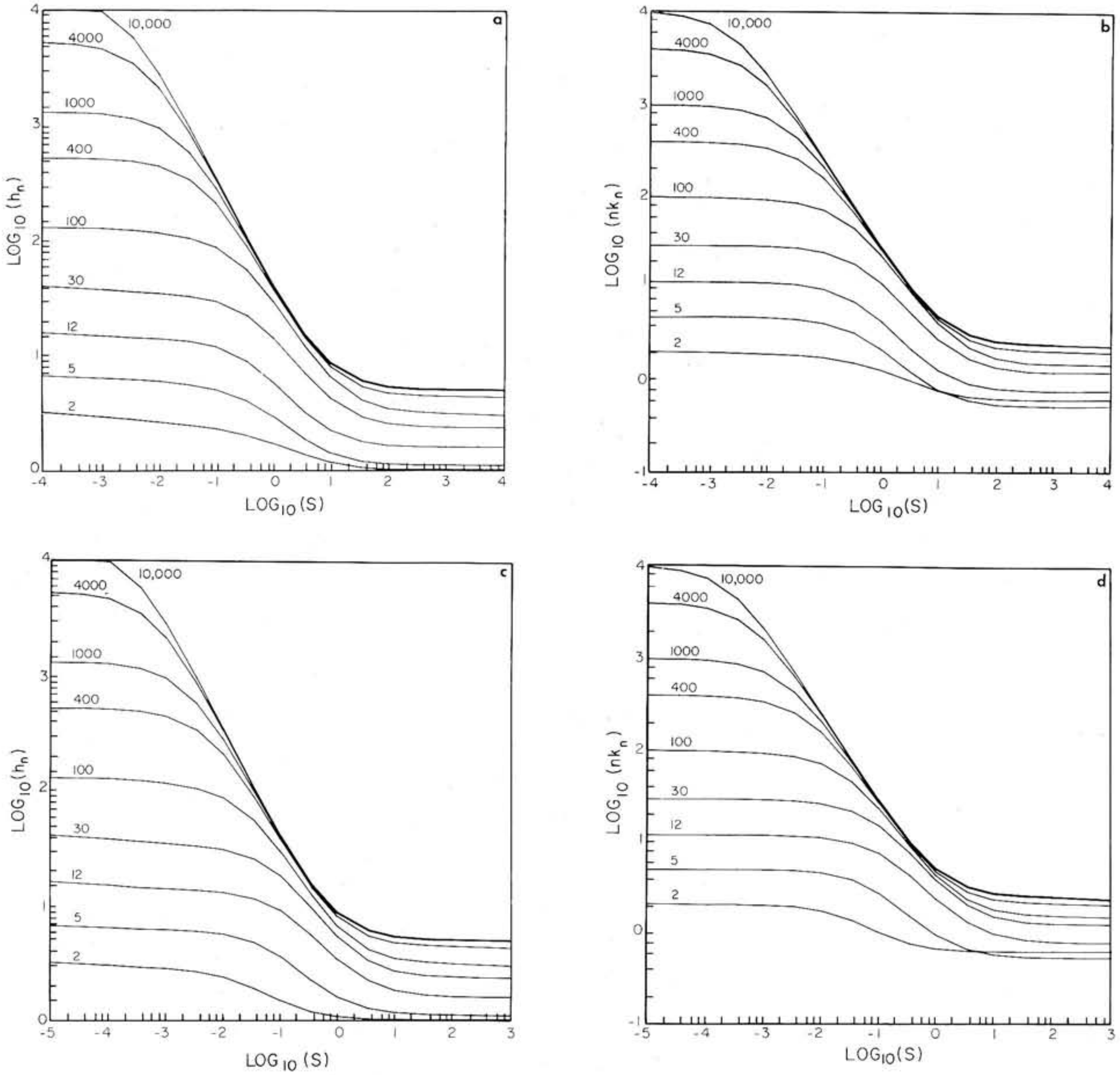

Fig. 11. Love number s spectra for (a) model 2 (h_n), (b) model 2 (nk_n), (c) model 3 (h_n), and (d) model 3 (nk_n).

center of maximum load, whether it is in Fennoscandia or in Canada, these curves inevitably show that the land has been emerging since deglaciation. This is completely in accord with the Green functions for radial displacement discussed above. However, at greater range this simple picture of land emergence since deglaciation is evidenced neither in the Green functions nor in the observations. For instance, the uplift curves along the Atlantic seaboard of North America become progressively more complicated along the profile from Newfoundland in the north to Florida in the south [*Walcott,* 1972]. The complex structure of these intermediate range response curves is entirely understandable in terms of the radial displacement Green functions. In these regions the response history is strongly dependent upon the shape and migration characteristics of the peripheral bulge. Figure 10 shows a series of slices at fixed θ through the radial displacement Green function for model 1. At sufficiently small angular separation from the load the height of land increases monotonically for all $t \leq$ 15,000 years. At larger separations this behavior changes to one in which the land first rises above and then sinks back toward the equilibrium level ($u_r \equiv 0$). At still greater range the height of land steadily decreases toward equilibrium.

The tilt Green functions for the three earth models discussed here are included in Figure 14(a–c). These functions are normalized with respect to the contribution from the direct attraction of the point mass load, which is just the first term on the right-hand side of (62):

$$t_{\text{norm}}(\theta) = \frac{-1}{m_e} \sum_{n=0}^{\infty} \frac{\partial P_n(\cos\theta)}{\partial\theta} = \frac{1}{m_e} \frac{\cos(\theta/2)}{4\sin^2(\theta/2)} \qquad (71)$$

Again, this normalization is the same as that employed by *Farrell* [1972] in his analysis of the elastic problem, and so the elastic part of the response discussed there is amenable to direct comparison with the viscous part of the response discussed here.

10. APPROACH TO ISOSTASY

In the preceding sections a discussion has been given of the response of a spherically stratified Maxwell medium to an impulsively applied point mass load. Although the time dependent Love numbers obtained for this problem and the

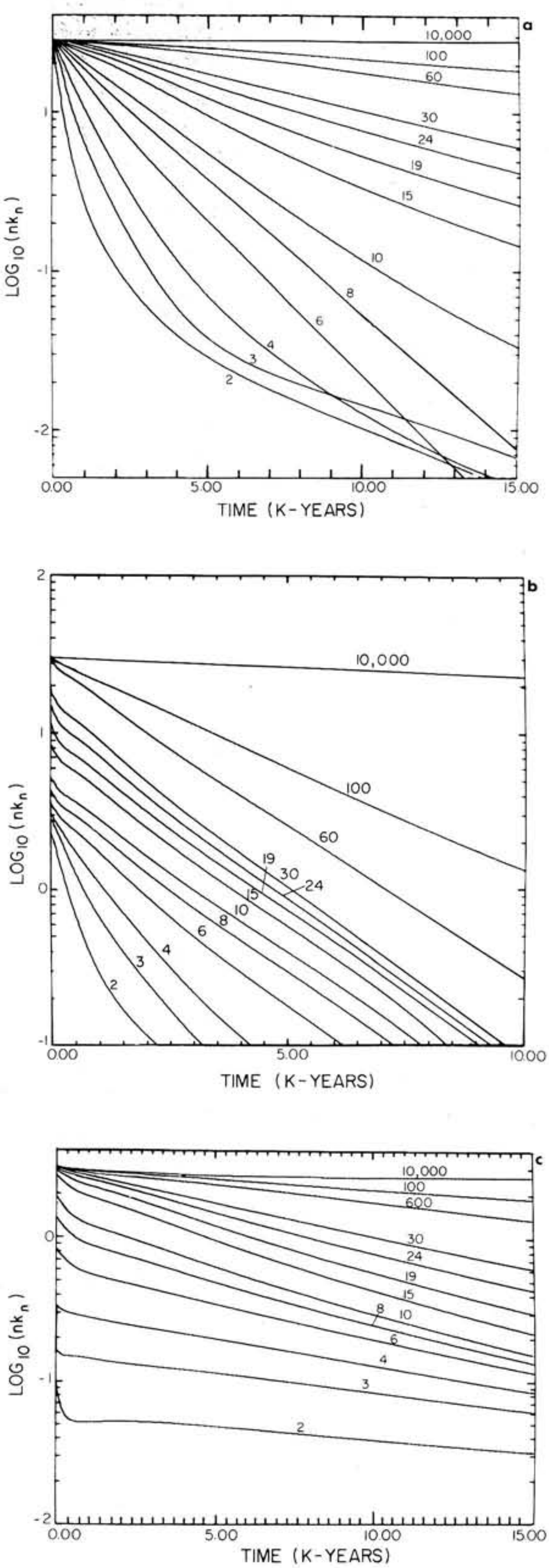

Fig. 12. Time dependent Love numbers nk_n for (*a*) model 1, (*b*) model 2, and (*c*) model 3.

Green functions derived from them were extremely useful in describing the response characteristics of the sphere, there remains an important property of the response that has yet to be described. This property concerns that ability of the models to pass smoothly from one equilibrium configuration to another in response to redistribution of the surface mass load. In such equilibrium states the surface mass load is entirely supported by the Archimedes, or buoyancy, force, and such states are termed states of isostatic equilibrium. A concise and clear review of the subject of isostasy has been given by *Garland* [1965]. The previous discussion of impulse response functions was concerned with the way in which the earth's shape was distorted and the way in which this distortion was erased when a surface point mass load was instantaneously applied and removed at $t = 0$. In this case the initial and final equilibrium states are the same and are states of no distortion of the initial spherical symmetry.

If the point mass had been allowed to remain on the surface for all $t \geq 0$, the system consisting of earth model plus mass load would have tended toward a new equilibrium configuration as $t \to \infty$. The impulse response functions obtained in the last section can be employed directly to determine the characteristics of this new isostatic equilibrium configuration. To determine, for instance, $u_r(\theta)$ in going from the old to the new equilibrium configuration simply requires convolution of (60) with the Heaviside function $H(t)$. The total response then consists of two parts: a time independent (immediate) elastic displacement with spherical harmonic coefficients ah_n^E/m_e and a time dependent part with spherical harmonic coefficients equal to a/m_e times the convolution of $h_n^V(t)$ with the Heaviside function. In the limit $t \to \infty$ the spherical harmonic coefficients of the latter part of the response tend to individual constants, since for sufficiently large t each of the $h_n^V(t)$ decays in an exponentiallike manner. This is obviously true for $h_n^V(t)$ approximated by Dirichlet series as was done in (54). As $t \to \infty$, the system tends to a new isostatic equilibrium. If the point mass is now moved from one point on the surface to another, the system will pass from this first equilibrium configuration to another. Such a redistribution of surface load is precisely the mechanism that leads to the relaxation processes observed to accompany deglaciation. The utility of the impulse response functions in describing the approach to isostatic equilibrium is thus clear.

It is useful to define a new set of isostatic Green functions derived from the impulse response functions of the last section by convolution with the Heaviside step function as discussed above and taking the limit $t \to \infty$. These functions are useful in that at least one of their number may be subjected to direct experimental measurement. This is the isostatic Green function for gravity anomaly. The extraction of this function from combined free air gravity anomaly and topographic data from the United States has been discussed by *Dorman and Lewis* [1970] and *Lewis and Dorman* [1970] without reference to any particular compensation mechanism. The function was later inverted [*Dorman and Lewis,* 1972] in conjunction with a particular assumption regarding the compensation mechanism to infer a radial dependence of density difference from a mean upper-mantle value. This isostatic Green function provides an additional constraint that any fluid model of the compensation mechanism must be able to satisfy.

11. CONCLUSIONS

The main purpose of the work discussed here has been to provide a new formalism with which to describe the phenomenon of postglacial 'uplift' in the context of a fluid (viscoelastic) model and in spherical geometry. The main advantage compared with previous treatments is obtained by direct application of the correspondence principle. This allows the description of the response in the Laplace transform domain through the use of the familiar equations of elasticity.

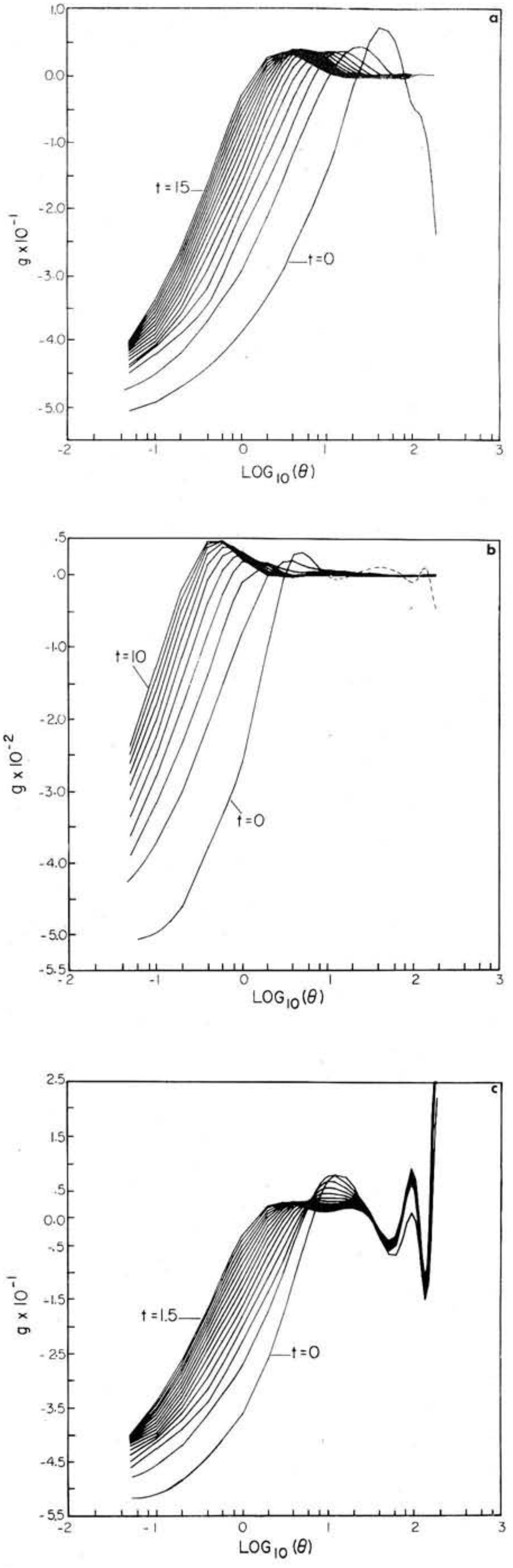

Fig. 13. Viscous part of the Green function for gravity anomaly in (*a*) model 1, (*b*) model 2, and (*c*) model 3.

The time domain behavior is then obtained by approximate Laplace inversion of a series of s spectra by using an extremal technique.

There are several rather general problems that are amenable to attack by using the methods developed here. Foremost among these is the direct testing of the fluid models of the compensation process in postglacial rebound. Given the time dependent and space dependent Pleistocene surface loads [*Paterson*, 1972], the Green functions derived in the previous sections may be employed to compute the time dependence of the changes produced in the earth's shape and in its external gravity field. This requires the evaluation of a space-time convolution integral. Both the viscous and the elastic parts of the response are of course to be included in this calculation. The computed response may then be compared with that which is observed in order to determine a single mantle viscosity profile of the model that is compatible with the observations. If it does not prove possible to provide a coherent explanation of all the observed data with such a model, then the model must be suitably altered, perhaps to incorporate the phase transition mechanism. All of this assumes that the Maxwell model of the interior is an adequate one. If the viscosity of the mantle is not Newtonian as *Post and Griggs* [1973] have suggested, then the Maxwell model of the interior is inappropriate. It is doubtful whether the postglacial uplift data are known with sufficient accuracy to enable such a distinction to be made, but this remains an open question.

If it is possible to find a single mantle viscosity profile that allows the Maxwell model to fit all the observations within a standard error, then the question of the uniqueness of this viscosity profile becomes a meaningful one. This question can be given a quantitative discussion by using the now widely employed technique developed by *Backus and Gilbert* [1967, 1968, 1970]. As might be expected, it turns out that the resolving kernels for this inverse problem have a particularly convenient representation in the Laplace transform domain. This fact makes the analysis of the rebound using the correspondence principle a particularly attractive way to proceed. A different approach to this problem was taken by *Parsons* [1972] in a discussion of the inverse problem for viscosity of the plane earth models employed by *McConnell* [1965]. The method based upon the techniques discussed here attacks the spherical problem without the necessity of making the usual approximation that each wavelength in the deformation relaxes exponentially. This approximation has been shown to be incorrect.

Given a viscosity model of known reliability, it will be possible to proceed toward the construction of iterative solutions to the nonlinear inverse problem in which the load distribution, as well as the mantle viscosity structure, is assumed indeterminate. When the viscosity model is fixed after one iteration, the load can be varied to obtain better agreement with the fine structure of the response. This process may be expected to yield particularly useful results for the Canadian arctic region, where the patterns of postglacial land emergence can be quite complicated [*Andrews*, 1966, 1968*a*, *b*].

APPENDIX: EQUATIONS OF MOTION

In (47) the matrix **A** has the following elements:

$$A_{1i} = [-2\lambda/\beta r,\ n(n+1)/\beta_r,\ 1/\beta,\ 0,\ 0,\ 0]$$

$$A_{2i} = (-1/r,\ 1/r,\ 0,\ 1/\mu,\ 0,\ 0)$$

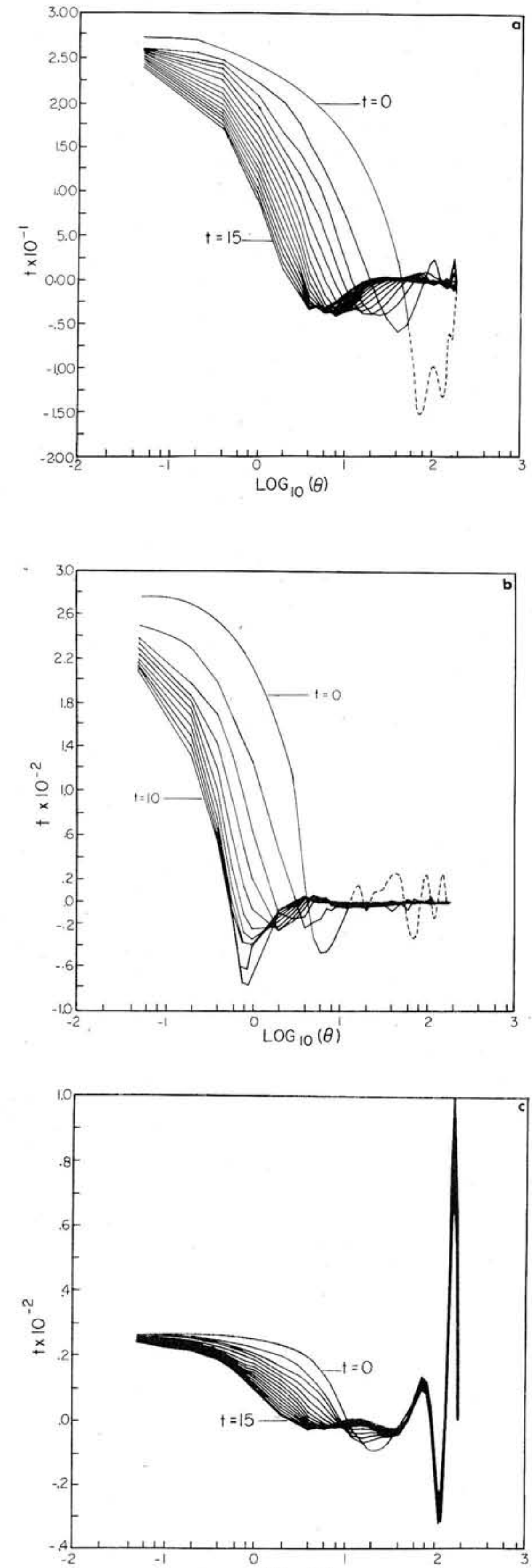

Fig. 14. Viscous part of the Green function for tilt in (*a*) model 1, (*b*) model 2, and (*c*) model 3.

$$A_{3i} = [(4/r)(\gamma/r - g_0\rho_0),\ (2\gamma/r - \rho_0 g_0)n(n+1)/\sigma,\ -4\mu/\beta r,\ n(n+1)/r,\ -\rho_0(n+1)/r,\ \rho_0]$$

$$A_{4i} = \{(1/r)(\rho_0 g_0 - 2\gamma/r),\ (-1/r^2)\cdot[-n(n+1)(\gamma+\mu)+2\mu],\ -\lambda/\beta r,\ -3/r,\ \rho_0/r,\ 0\}$$

$$A_{5i} = [-r\pi G\rho_0,\ 0,\ 0,\ 0,\ -(n+1)/r,\ 1]$$

$$A_{6i} = [-(n+1)4\pi G\rho_0/r,\ n(n+1)4\pi G\rho_0/r,\ 0,\ 0,\ 0,\ (n+1)/r]$$

where $\gamma = \mu(3\lambda + 2\mu)/(\lambda + 2\mu)$, $\beta = \lambda + 2\mu$, and λ and μ are the s dependent forms given in (7).

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REFERENCES

Alterman, Z., H. Jarosch, and C. L. Pekeris, Propagation of Rayleigh waves in the earth, *Geophys. J., 4,* 219, 1961.

Andrews, J. T., Pattern of coastal uplift and deglaciation, west Baffin Island, Northwest Territories, *Geogr. Bull., 8*(2), 174, 1966.

Andrews, J. T., Postglacial rebound in arctic Canada: Similarity and prediction of uplift curves, *Can. J. Earth Sci., 5,* 39, 1968*a*.

Andrews, J. T., Pattern and cause of variability of postglacial uplift and rate of uplift in arctic Canada, *J. Geol., 76,* 404, 1968*b*.

Artyushkov, E. V., and Yu. A. Mescherikov, Recent movements of the earth's crust and isostatic compensation, in *The Earth's Crust and Upper Mantle, Geophys. Monogr. Ser.,* vol. 13, edited by P. J. Hart, p. 379, AGU, Washington, D. C., 1969.

Backus, G. E., Converting vector and tensor equations to scalar equations in spherical co-ordinates, *Geophys. J., 13,* 71, 1967.

Backus, G. E., and J. F. Gilbert, Numerical applications of a formalism for geophysical inverse problems, *Geophys. J. Roy. Astron. Soc., 13,* 247, 1967.

Backus, G. E., and J. F. Gilbert, The resolving power of gross earth data, *Geophys. J. Roy. Astron. Soc., 16,* 169, 1968.

Backus, G. E., and J. F. Gilbert, Uniqueness in the inversion of inaccurate gross earth data, *Phil. Trans. Roy. Soc. London, Ser. A, 266,* 123, 1970.

Bellman, R., R. E. Kalaba, and J. A. Lockett, *Numerical Inversion of the Laplace Transform,* Elsevier, New York, 1966.

Biot, M. A., Theory of stress-strain relations in anisotropic visco elasticity and relaxation phenomena, *J. Appl. Phys., 25*(11), 1385, 1954.

Biot, M. A., Variational principles in irreversible thermodynamics with application to visco elasticity, *Phys. Rev., 97,* 1463, 1955*a*.

Biot, M. A., Dynamics of visco elastic anisotropic media, in *Proceedings of the Second Midwestern Conference on Solid Mechanics, Res. Ser.,* vol. 129, Engineering Experiment Station, Purdue University, Lafayette, Ind., 1955*b*.

Boussinesq, J., Application des potentiels a l'étude de l'équilibre et du mouvement des solides élastiques, Gauthier-Villars, Paris, 1885.

Busse, F., and G. Schubert, Convection in a fluid with two phases, *J. Fluid Mech., 46,* 801, 1971.

Cathles, L. M., The viscosity of the earth's mantle, Ph.D. thesis, Princeton Univ., Princeton, N. J., 1971.

Chandrasekhar, S., *Ellipsoidal Figures of Equilibrium, Silliman Lecture Ser.,* vol. 41, Yale University Press, New Haven, Conn., 1970.

Crittenden, M. D., Jr., Effective viscosity of the earth derived from isostatic loading of Pleistocene Lake Bonneville, *J. Geophys. Res., 68,* 5517, 1963.

Crittenden, M. D., Jr., Viscosity and finite strength of the mantle as determined from water and ice loads, *Geophys. J. Roy. Astron. Soc., 14,* 261, 1967.

Dahlen, F. A., On the static deformation of an earth model with a fluid core, *Geophys. J. Roy. Astron. Soc., 34,* 1, 1974.

Daly, R. A., Oscillations of level in the belts peripheral to the Pleistocene ice caps, *Geol. Soc. Amer. Bull., 31,* 303, 1920.

Darwin, G. H., On the bodily tides of viscous and semi-elastic spheroids, and on the ocean tides upon a yielding nucleus, *Phil. Trans. Roy. Soc. London, Ser. A, 1970,* 1, 1879.

Dicke, R. H., The secular acceleration of the earth's rotation, in *The Earth-Moon System,* edited by B. G. Marsden and A. G. W. Cameron, p. 98, Plenum, New York, 1966.

Dicke, R. H., Average acceleration of the earth's rotation and the viscosity of the deep mantle, *J. Geophys. Res., 74,* 5895, 1969.

Dorman, L. M., and B. T. R. Lewis, Experimental isostasy, 1, Theory of the determination of the earth's isostatic response to a concentrated load, *J. Geophys. Res., 75,* 3357, 1970.

Dorman, L. M., and B. T. R. Lewis, Experimental isostasy, 3, Inversion of the isostatic changes, *J. Geophys. Res., 77,* 3068, 1972.

Eringen, A. C., *Mechanics of Continua,* John Wiley, New York, 1967.

Farrell, W. E., Deformation of the earth by surface loads, *Rev. Geophys. Space Phys., 10,* 761, 1972.

Farrell, W. E., Earth tides, ocean tides, and tidal loading, *Phil. Trans. Roy. Soc. London, Ser. A, 274,* 45, 1973.

Gaposchkin, E. M., and K. Lambeck, Earth's gravity field to sixteenth degree and station coordinates from satellites and terrestrial data, *J. Geophys. Res., 76,* 4855, 1971.

Garland, G. D., *The Earth's Shape and Gravity,* Pergamon, New York, 1965.

Gilbert, F., and G. Backus, Elastic-gravitational vibrations of a radially stratified sphere, in *Dynamics of Stratified Solids,* edited by G. Herrmann, p. 827, American Society of Mechanical Engineers, New York, 1968.

Gjevik, B., Surface readjustment owing to a subcrustal phase transition, *Phys. Earth Planet. Interiors, 5,* 403, 1972.

Gjevik, B., The dynamical response of a subcrustal phase boundary to pressure perturbations, *Phys. Earth Planet. Interiors, 7,* 401, 1973.

Goldreich, P., and A. Toomre, Some remarks on polar wandering, *J. Geophys. Res., 74,* 2555, 1969.

Haskell, N. A., The motion of a viscous fluid under a surface load, *Physics, 6,* 265, 1935.

Haskell, N. A., The motion of a viscous fluid under a surface load, 2, *Physics, 7,* 56, 1936.

Haskell, N. A., The viscosity of the asthenosphere, *Amer. J. Sci., 33,* 22, 1937.

Herring, C., Diffusional viscosity of a polycrystalline solid, *J. Appl. Phys., 21,* 437, 1950.

Higgins, G., and G. C. Kennedy, The adiabatic gradient and the melting point gradient in the core of the earth, *J. Geophys. Res., 76,* 1870, 1971.

Hildebrand, F. B., *Introduction to Numerical Analysis,* McGraw-Hill, New York, 1956.

Innes, M. J. S., and A. Weston, Crustal uplift of the Canadian shield and its relation to the gravity field, *Ann. Acad. Sci. Fenn., Ser. A3,* 169, 1966.

Jamieson, T. F., *Geol. Mag., 9,* 461, 1882.

Jeffreys, H., *Geogr. J., 95,* 384, 1940.

Jeffreys, H., *The Earth,* 4th ed., Cambridge University Press, London, 1970.

Kaula, W. M., Elastic models of the mantle corresponding to variations in the external gravity field, *J. Geophys. Res., 68,* 4967, 1963.

Kaula, W. M., Global gravity and tectonics, in *The Nature of the Solid Earth,* edited by E. C. Robertson, p. 385, McGraw-Hill, New York, 1972.

Kuo, J. T., Static response of a multilayered medium under inclined surface loads, *J. Geophys. Res., 74,* 3195, 1969.

Lee, E. H., Stress analysis in visco-elastic bodies, *Quart. Appl. Math., 13,* 183, 1955.

Lee, E. H., and T. G. Rogers, Solution of visco-elastic stress analysis problems using measured creep and relaxation functions, *Interim Tech. Rep. 1,* project 1892-E, Brown University, Providence, R. I., 1961.

Lewis, B. T. R., and L. M. Dorman, Experimental isostasy, 2, An isostatic model of the United States derived from gravity and topographic data, *J. Geophys. Res., 75,* 3367, 1970.

Longman, I. M., A Green's function for determining the deformation of the earth under surface mass loads, 1, Theory, *J. Geophys. Res., 67,* 845, 1962.

Longman, I. M., A Green's function for determining the deformation of the earth under surface mass loads, 2, Computations and numerical results, *J. Geophys. Res., 68,* 485, 1963.

MacDonald, G. J. F., The figure and long term mechanical properties of the earth, in *Advances in Earth Sciences,* edited by P. M. Hurley, MIT Press, Cambridge, Mass., 1966.

Magnitsky, V. A., and I. V. Kalashnikova, Problem of phase transitions in the upper mantle and its connection with the earth's crustal structure, *J. Geophys. Res., 75,* 377, 1970.

McConnell, R. K., The visco elastic response of a layered earth to the removal of the Fennoscandian ice sheet, Ph.D. thesis, Univ. of Toronto, Toronto, Ont., 1963.

McConnell, R. K., Isostatic adjustment in a layered earth, *J. Geophys. Res., 70,* 5171, 1965.

McConnell, R. K., Viscosity of the mantle from relaxation time spectra of isostatic adjustment, *J. Geophys. Res., 73,* 7089, 1968*a*.

McConnell, R. K., Viscosity of the earth's mantle, in *The History of the Earth's Crust,* edited by R. A. Phinney, p. 45, Princeton University Press, Princeton, N. J., 1968*b*.

McKenzie, D. P., The viscosity of the lower mantle, *J. Geophys. Res., 71,* 3995, 1966.

McKenzie, D. P., The viscosity of the mantle, *Geophys. J. Roy. Astron. Soc., 14,* 297, 1967.

McKenzie, D. P., The geophysical importance of high temperature creep, in *The History of the Earth's Crust,* edited by R. A. Phinney, Princeton University Press, Princeton, N. J., 1968.

McKenzie, D. P., J. M. Roberts, and N. O. Weiss, Convection in the earth's mantle: Towards a numerical simulation, *J. Fluid Mech., 62,* 465, 1974.

Muki, R., and E. Sternberg, On transient thermal stresses in visco elastic materials with temperature dependent properties, *J. Appl. Mech., 29,* 193, 1961.

Munk, W. H., and G. J. F. MacDonald, *The Rotation of the Earth,* Cambridge University Press, New York, 1960.

O'Connell, R. J., Pleistocene glaciation and the viscosity of the lower mantle, *Geophys. J. Roy. Astron. Soc., 23,* 299, 1971.

Parsons, B. E., Changes in the earth's shape, Ph.D. thesis, Cambridge Univ., London, 1972.

Paterson, W. S. B., Laurentide ice sheet: Estimated volumes during late Wisconsin, *Rev. Geophys. Space Phys., 10,* 885, 1972.

Pekeris, C. L., and Y. Accad, Dynamics of the liquid core of the earth, *Phil. Trans. Roy. Soc. London, Ser. A, 273,* 19, 1972.

Peltier, W. R., Thermal stability of non-Boussinesq configurations, Ph.D. thesis, Univ. of Toronto, Toronto, Ont., 1971.

Peltier, W. R., Penetrative convection in the planetary mantle, *Geophys. Fluid Dyn., 3,* 265, 1972.

Pertsev, B. P., On the effect of ocean tides on tidal variations of gravity, *Izv. Acad. Sci. USSR Phys. Solid Earth,* Engl. Transl., *10,* 25, 1966.

Pertsev, B. P., The effect of ocean tides upon earth tide observations, in *Sixth International Symposium on Earth Tides,* edited by R. Dejaiffe, p. 113, Royal Observatory of Belgium, Brussels, 1970.

Post, R. L., and D. T. Griggs, The earth's mantle: Evidence of non-linear Newtonian flow, *Science, 181,* 1242, 1973.

Ralston, A., and H. S. Wilf, *Mathematical Methods for Digital Computers,* John Wiley, New York, 1960.

Richter, F. M., Dynamic models for sea floor spreading, *Rev. Geophys. Space Phys., 11,* 223, 1973.

Ringwood, A. E., Phase transformations and the constitution of the mantle, *Phys. Earth Planet. Interiors, 3,* 109, 1970.

Ringwood, A. E., Mineralogy of the deep mantle: Current status and future developments, in *The Nature of the Solid Earth,* edited by E. C. Robertson, McGraw-Hill, New York, 1972.

Ringwood, A. E., and A. Major, The system Mg_2SiO_4 at high pressures and temperatures, *Phys. Earth Planet. Interiors, 3,* 89, 1970.

Schapery, R. A., Two simple approximate methods of Laplace transform inversion for visco-elastic stress analysis, *Rep. SM 61-23,* Grad. Aeronaut. Lab., Calif. Inst. of Technol., Pasadena, 1961*a*.

Schapery, R. A., Approximate methods of transform inversion for visco-elastic stress analysis, in *Proceedings of the Fourth U.S. National Congress of Applied Mechanics,* p. 1075, American Society of Mechanical Engineers, New York, 1961*b*.

Schapery, R. A., Irreversible thermodynamics and variational principles with applications to visco-elasticity, Ph.D. thesis, Calif. Inst. of Technol., Pasadena, 1962.

Scheidegger, A. E., Rheology of the earth: The basic problem of geodynamics, *Can. J. Phys., 35,* 383, 1957.

Scheidegger, A. E., *Principles of Geophysics,* Springer, Berlin, 1963.

Schubert, G., and D. L. Turcotte, Phase changes and mantle convection, *J. Geophys. Res., 76,* 1424, 1971.

Shanks, E. B., Solutions of differential equations by evaluations of functions, *Math. Comput., 20,* 21, 1966.

Smilie, D. E., and L. Mansinha, The elasticity theory of dislocations in real earth models, *Geophys. J. Roy. Astron. Soc., 23,* 329, 1971.
Takeuchi, H., M. Saito, and N. Kobayashi, Statical deformations and free oscillations of a model earth, *J. Geophys. Res., 67,* 1141, 1962.
Tozer, D. C., Heat transfer and convection currents, *Phil. Trans. Roy. Soc. London, Ser. A, 258,* 252, 1965.
Turcotte, D. L., and G. Schubert, Structure of the olivine-spinel phase boundary in the descending lithosphere, *J. Geophys. Res., 76,* 7980, 1971.
Turcotte, D. L., and G. Schubert, Structure of the olivine-spinel phase boundary in the descending lithosphere, *J. Geophys. Res., 77,* 2146, 1972.
Walcott, R. I., Isostatic response to loading of the crust in Canada, *Can. J. Earth Sci., 7,* 716, 1970.
Walcott, R. I., Late quaternary vertical movements in eastern North America: Quantitative evidence of glacio-isostatic rebound, *Rev. Geophys. Space Phys., 10,* 849, 1972.
Weertman, J., Theory of steady state creep based upon dislocation climb, *J. Appl. Phys., 26,* 1213, 1955.
Weertman, J., The creep strength of the earth's mantle, *Rev. Geophys. Space Phys., 8,* 145, 1970.

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