Oceanic Mean Flows Forced by Dissipating Topographic Internal Waves

Nicolas Grisouard, Oliver Bühler
New York University - Courant Institute of Mathematical Sciences
grisouard@cims.nyu.edu, obuhler@cims.nyu.edu

NEW YORK UNIVERSITY
NSF/OCE 1024180
NSF/DMS 1009213

Motivation

- Internal waves (IWs) in the ocean: important for the large-scale circulation AND too small for GCMs.
- Classical IW effect on atmospheric mean (i.e. zonal) flows:
  a zonal mean flow "pushes" a mountain → internal (lee) waves → dissipation → drag on mean flow...
  … an action-reaction pair, except **reaction is felt by the mean flow where waves are dissipated!**
- Why is atmospheric knowledge so hard to apply to the ocean? Because **absence of zonal symmetry** and **random forcing** imply that (i) **processes are 3D in nature** and (ii) a new average has to be used.
- We focus on the most energetic component of the IW forcing spectrum: the semi-diurnal tide.

Can dissipating internal waves radiated from the tide-topography interaction force significant mean flows?

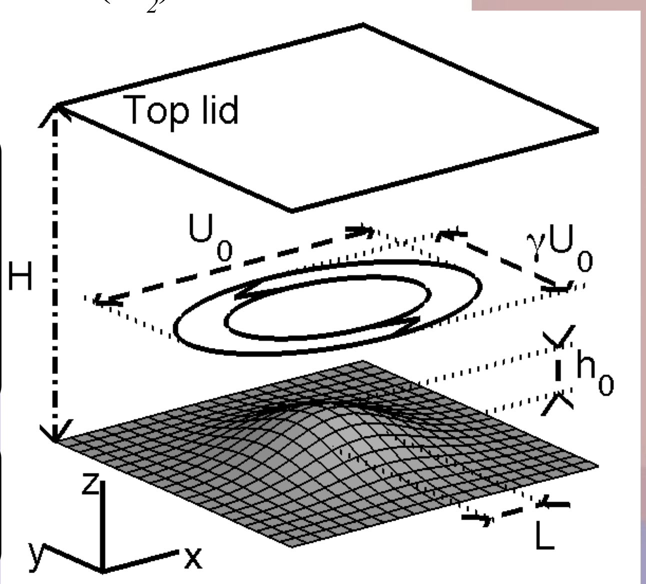
Physical and Mathematical Set-up

- Rotating ocean: depth H=4 km, Coriolis frequency $f = 10^{-4}$ s^{-1} (45º), constant buoyancy frequency $N = 10^{-3}$ s^{-1}
- Topography at $z = h(\boldsymbol{r})$, max. elevation h_0=100 m, horizontal scale L=10 km
- Elliptic barotropic tide $\boldsymbol{U}$(t): amplitude U_0=1.4 cm/s, frequency $\omega=1.4\times10^{-4}$ s^{-1} (M_2)
- Internal waves slope: $\mu=\sqrt{(\omega^2-f^2)/(N^2-\omega^2)}\approx 0.1$

Weak topography: $a=\dfrac{h_0}{\mu L}\ll 1$

- Bottom BC applied at $z = 0$ ($\neq h(\boldsymbol{r})$)
- Asymptotic ordering of solutions:
  - $O(1)$: barotropic tide
  - $O(a)$: linear IWs radiated by topography
  - $O(a^2)$: mean flow generated by linear IWs

Small tidal excursion: $\epsilon=\dfrac{U_0}{\omega L}\ll 1 \;\Rightarrow\; \forall\phi=O(a),\;\; \phi(\boldsymbol{r},z,t)=\Re\left[\tilde\phi(\boldsymbol{r},z)\,\mathrm{e}^{-i\omega t}\right]$



$$\nabla\cdot\boldsymbol{u}=0,$$
$$\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+f\,\hat{\boldsymbol{z}}\times\boldsymbol{u}=-\nabla p+b\,\hat{\boldsymbol{z}},$$
$$\partial_t b+(\boldsymbol{u}\cdot\nabla)b+N^2w=\underline{-\alpha\,b},$$
$$w|_{z=0}=(\boldsymbol{U}+\boldsymbol{u})\cdot\nabla_h h(\boldsymbol{r}),\qquad w|_{z=H}=0.$$

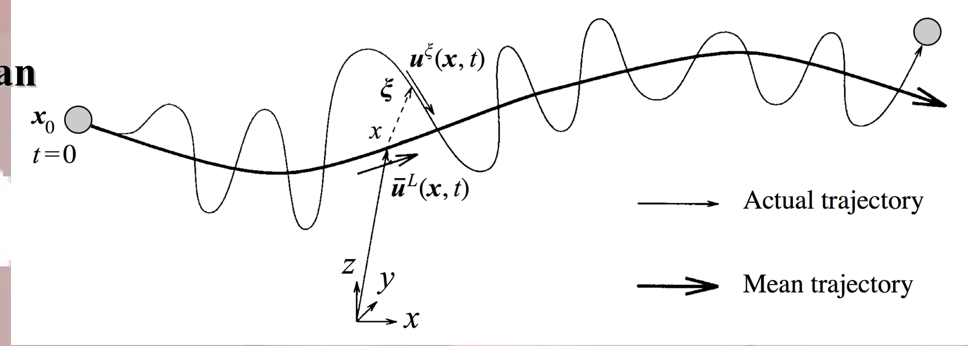
- **Boussinesq equations** in the tidal reference frame
- **Radiative damping** (simple + conserves momentum)
- Vertical BCs: no-normal flow with weak topography
- All horizontal fluxes are outward + decay at infinity

$\boldsymbol{u} = (u,v,w)$: velocity departure from the basic tide
$b = -g\rho'/\rho_0$: buoyancy p: scaled pressure

$O(a^2)$: Mean Flow Forcing

- We **average over fast time scales** (say ω^{-1}) and let the mean flow evolve slowly.
- The linear Boussinesq operator accepts one ω=0, balanced mode which (i) does not appear at $O(a)$ but can be forced at $O(a^2)$ and (ii) is the only mode captured by the fast time average. **Can the $O(a^2)$ balanced mode be forced resonantly by the $O(a)$ IWs?**
- We choose to use the **Generalized Lagrangian-Mean (GLM) theory**, hybrid between Eulerian and Lagrangian:

$$\forall\phi,\;\; \phi^\xi(\boldsymbol{x},t)\overset{\text{def}}{=}\phi(\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x},t),t),$$
$$\phi^\xi=\bar\phi^L+\phi^l,\quad\text{where}\quad \bar\phi^L\overset{\text{def}}{=}\overline{\phi^\xi}\quad\text{and therefore}\quad \overline{\phi^l}=0.$$



- GLM equations in vorticity form, taking into account $\partial_t\overline{\text{wave quantities}} = 0$ for our steady fields:

$$\nabla\cdot\bar{\boldsymbol{u}}^L=0,$$
$$\partial_t(\nabla\times\bar{\boldsymbol{u}}^L)-f\,\partial_z\bar{\boldsymbol{u}}^L-\nabla\times(\bar b^L\hat{\boldsymbol{z}})=\nabla\times\boldsymbol{F},\qquad \boldsymbol{F}=\frac{\alpha N^2}{2(\alpha^2+\omega^2)\omega}\Im(\tilde w^*\nabla\tilde w)$$
$$\partial_t\bar b^L+N^2\bar w^L=-\alpha^L\bar b^L,\quad 0\leqslant\alpha^L\leqslant\alpha$$

mean flow might not be dissipated as much as the waves!

- Lagrangian-mean potential vorticity evolution equation: $\dfrac{\partial\bar Q^L}{\partial t}+\dfrac{\alpha^L f}{N^2}\partial_z\bar b^L=(\nabla\times\boldsymbol{F})\cdot\hat{\boldsymbol{z}},$
- **Is this forcing weak or strong?** (i.e. steady state at $O(a^2)$ vs. resonant growth)
  (i) If one enforces $\partial_t = \partial_y = 0$ in the GLM equations, one finds out that the forcing is weak:

  No mean flow can be forced resonantly in a vertical 2D configuration.

  (ii) From the PV equation: a mean flow can grow resonantly in 3D if $\alpha^L f/N^2\rightarrow 0$ …

  … and if not, $\int(\nabla\times\boldsymbol{F})\cdot\hat{\boldsymbol{z}}\,\mathrm{d}z=-f\left(\bar w^L|_{z=H}-\bar w^L|_{z=0}\right)$ is not obviously true anyway!

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$O(a)$: Linear Internal Waves

- Equations to solve: equation for the vertical velocity + vertical BCs:

$$\nabla_h^2\tilde w+(\mu\beta)^2\partial_{zz}\tilde w=0,\quad \tilde w|_{z=H}=0,\quad \tilde w|_{z=0}=\tilde{\boldsymbol{U}}\cdot\nabla_h h(\boldsymbol{r}),\quad \beta^2=fct(\alpha,N,\omega)\in\mathbb{C},\;\; \beta=1 \text{ if } \alpha=0.$$

- We look for a **Green's function whose point source is on the bottom**:

$$\nabla_h^2G+(\mu\beta)^2\partial_{zz}G=0,\quad G|_{z=H}=0,\quad G|_{z=0}=\delta(\boldsymbol{r}-\boldsymbol{r_0}).$$

- Possible candidate: $G(\boldsymbol{r},z;\boldsymbol{r_0})=\sum_{m\in\mathbb{N}}G^m(\boldsymbol{r};\boldsymbol{r}_0)\sin(m\pi z/H)+(1-z/H)\delta(\boldsymbol{r}-\boldsymbol{r}_0),$

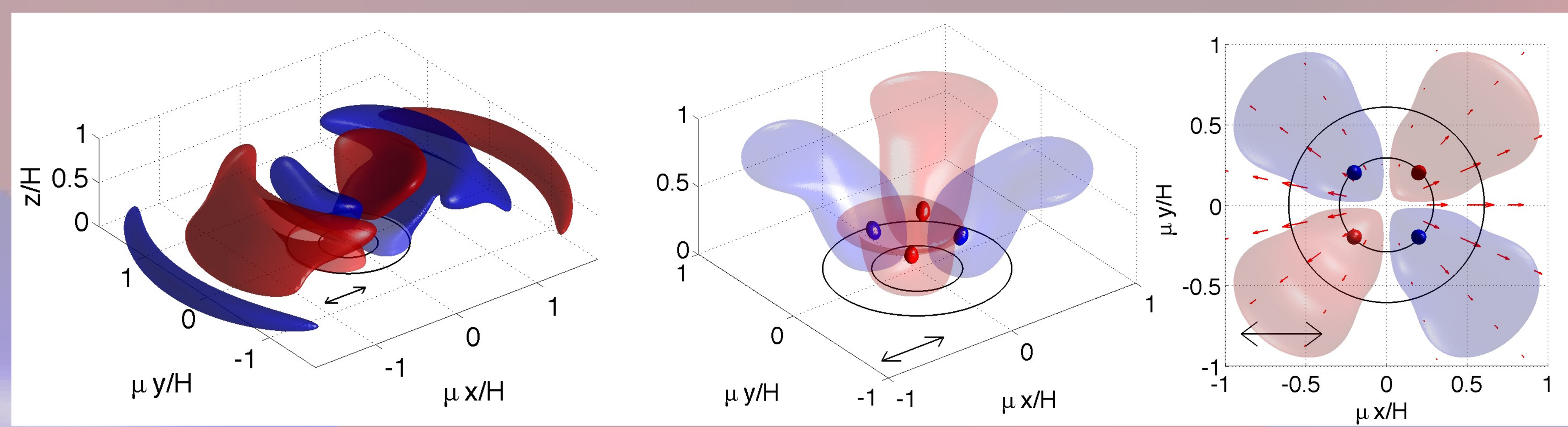
$$\nabla_h^2G^m+(\mu\beta m\pi/H)^2G^m=-2\delta(\boldsymbol{r}-\boldsymbol{r_0})/(m\pi),\qquad G^m|_{z=H}=G^m|_{z=0}=0$$

- Each G^m is the solution of a **forced 2D Helmholtz equation**; with the outward radiation condition, its Green's function ("Green's function of the Green's function") is known: $g^m(\boldsymbol{r_0};\boldsymbol{r}')=-(i/4)H_0^{(1)}(\mu\beta m\pi|\boldsymbol{r}'-\boldsymbol{r}_0|/H)\quad(H_0^{(1)}=J_0+iY_0)$
- Convolving $g^m(\boldsymbol{r}_0;\boldsymbol{r}')$ with $-2\delta(\boldsymbol{r}-\boldsymbol{r}_0)/(m\pi)$ gives the G^ms, which give G; convolving G with the bottom BC finally gives:

$$\tilde w(\boldsymbol{r},z)=\frac{(\mu\beta)^2\pi}{2iH^2}\sum_{m=1}^{\infty}C^m(\boldsymbol{r})\sin\left(\frac{m\pi z}{H}\right)+\left[1-\frac{z}{H}+\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\frac{m\pi z}{H}\right)\right]\tilde w|_{z=0}(\boldsymbol{r}),$$
$$\text{with } C^m(\boldsymbol{r})=\iint_{\mathbb{R}^2}m\,H_0^{(1)}(\mu\beta m\pi|\boldsymbol{r}-\boldsymbol{r_0}|/H)\,\tilde w|_{z=0}(\boldsymbol{r_0})\,\mathrm{d}^2\boldsymbol{r_0}.$$

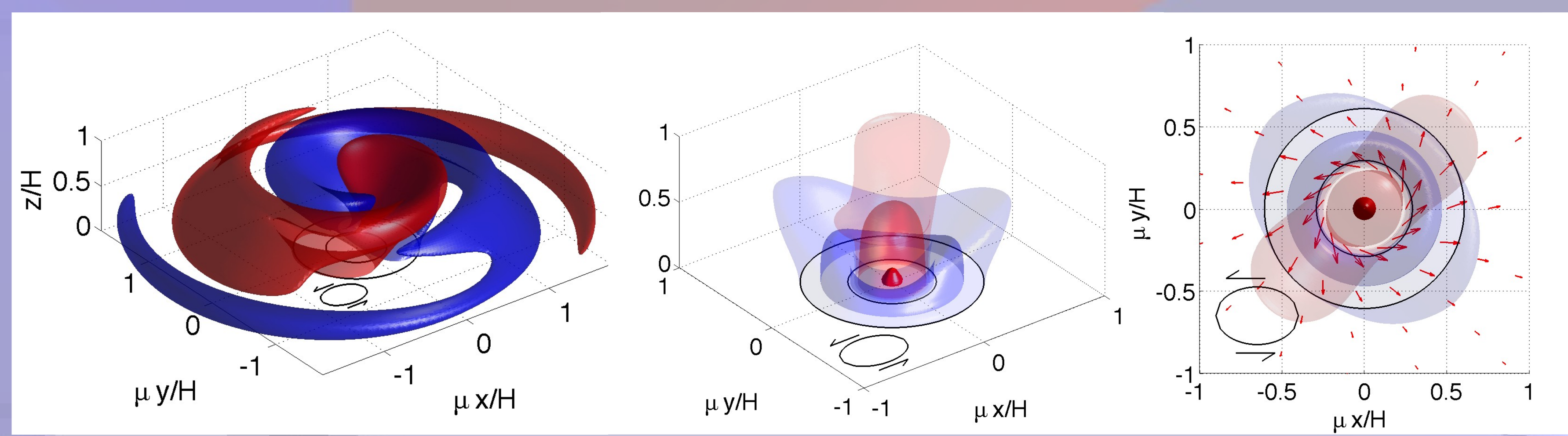
Mean PV forcing: Numerical Calculations

- Gaussian bump, $\gamma = 0$: waves emitted in two opposite directions ⇒ $\boldsymbol{F}$ in two opposite directions, aligned with the tide ⇒ $(\nabla\times\boldsymbol{F})\cdot\hat{\boldsymbol{z}}$ **quadrupolar** (no net force acting on the mountain, nor on the fluid).
  With $\alpha^{-1}=O$(week), spin-up time of the balanced flow is O(years).



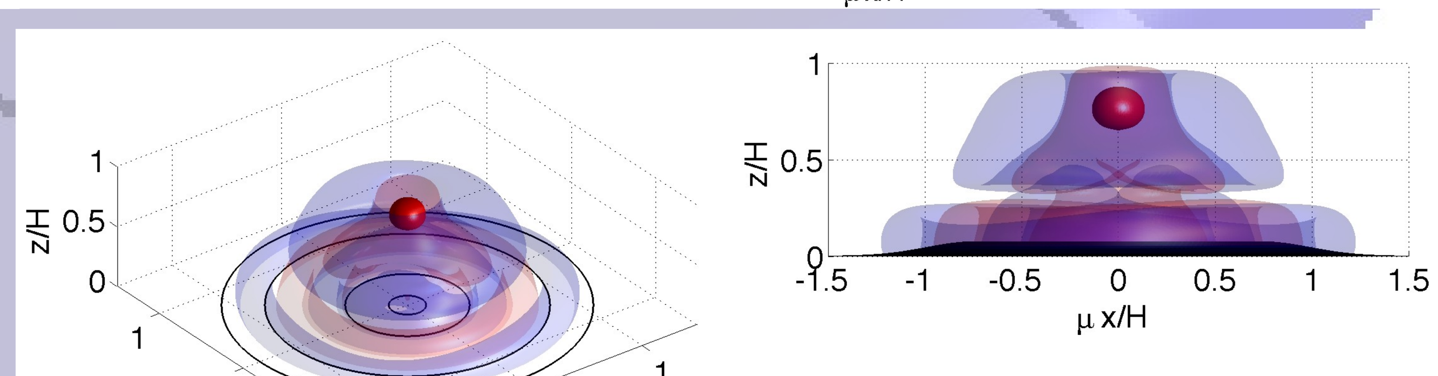
Left: contours of the O(a) vertical velocity field. Black arrow: direction of the tide; circles: mountain.
Middle and right: contours of $(\nabla\times\boldsymbol{F})\cdot\hat{\boldsymbol{z}}$; red arrows mark the direction of $\boldsymbol{F}$.

- Gaussian bump, $\gamma = 0.7$: **quadrupolar pattern vanishes and a topography-trapped vortex takes over**, counterclockwise for $\gamma > 0$. Max. amplitude of $(\nabla\times\boldsymbol{F})\cdot\hat{\boldsymbol{z}}$: 13 times stronger than for $\gamma = 0$.



Same as above for $\gamma = 0.7$.
Black ellipse: aspect ratio and rotational direction of the tide

- Gaussian annulus topography, $\gamma = 0.7$: IWs focus high above the topography, where the PV forcing is strongest. This situation could happen above flat-top seamounts.



Left: contours of $(\nabla\times\boldsymbol{F})\cdot\hat{\boldsymbol{z}}$.
Right: same as left, viewed from the side.

Conclusions and Perspectives

- In an all-small parameter regime, we derived a 3D internal tide radiation model that picks up the BCs properly.
- We computed the effective force due to the dissipating waves, acting on the Lagrangian-mean, balanced flow, and noticed that only 3D configurations can generate significant mean flows.
- In the case of a rectilinear tide, the effective force acts in two opposite directions, aligned with the tide, corresponding to a quadrupolar pattern for the PV forcing. In the case of an elliptic tide, topography-trapped vortices are forced, whose rotational directions depend on the one of the tide.
- Perspectives: (i) implement Laplacian (or any other momentum-conserving) dissipation operator, (ii) release the smallness assumptions in order to investigate realistic settings (Guyots, steep ridges...), (iii) investigate the long-term behavior of the PV via numerical simulations...